Odd symmetric functions and categorification

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ABSTRACT

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We introduce q- and signed analogues of several constructions in and around the theory of symmetric functions. The most basic of these is the Hopf superalgebra of odd symmetric functions. This algebra is neither (super-)commutative nor (super-)cocommutative, yet its combinatorics still exhibit many of the striking integrality and positivity properties of the usual symmetric functions. In particular, we give odd analogues of Schur functions, Kostka numbers, and Littlewood-Richardson coefficients.

Using an odd analogue of the nilHecke algebra, we give a categorification of the integral divided powers form of $U_q^+(\mathfrak{sl}_2)$ inequivalent to the one due to Khovanov-Lauda. Along the way, we develop a graphical calculus for indecomposable modules for the odd nilHecke algebra.

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Acknowledgments

Rather than attempt to unpack the complicated and nonlinear system of people, places, and things that have enabled me to write this thesis, I will err on the side of omission.

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Introduction

Symmetric functions and categorification

A polynomial or power series in variables $x_1, \ldots, x_n$ is called *symmetric* if any permutation, acting on subscripts, leaves that polynomial invariant. The algebra Λ of symmetric functions appear all over mathematics—the representation theory of symmetric groups, the representation theory of $GL_n(\mathbb{C})$, probability, combinatorics, enumerative geometry, and algebraic topology, to name a few.

As an object which appears in so many areas, Λ naturally carries many types of structure. A selection:

- a Hopf algebra structure: multiplication, comultiplication, bialgebra axiom;
- a bilinear form which makes multiplication and comultiplication adjoint linear maps;
- bases with $\mathbb{Z}_{\geq 0}$-valued structure coefficients and on which the bilinear form is $\mathbb{Z}_{\geq 0}$-valued.

These comprise *prima facie* evidence that Λ can be *categorified*—that there exists a graded additive category $\mathcal{C}$ such that $K_0(\mathcal{C}) \cong \Lambda$ (split Grothendieck group) with the following structures:

- functors $F_m : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $F_\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ which descend to multiplication and comultiplication on K_0;
- an adjunction of functors $F_m \dashv F_\Delta$;

- collections of objects whose images in K_0 are the special bases of Λ and whose hom-space dimensions match the values of the bilinear form on these elements, such that decomposition numbers of images under F_m match the structure coefficients.

In other words, categorification replaces an algebra with structure by a category with higher structure, all in a nicely compatible way.

For example, suppose $\{V_i\}_{i\in I}$ is the set of indecomposable objects of an additive category $\mathcal{C}$ as above. The set $\{v_i = [V_i]\}_{i\in I}$ is a basis for $K_0(\mathcal{C})$. If there is an isomorphism $F_m(V_i, V_j) \cong \bigoplus_{k\in I} V_k^{\oplus a_{ij}^k}$ for certain non-negative integers a_{ij}^k, then on $K_0(\mathcal{C})$, this becomes the equation $v_i v_j = \sum_{k\in I} a_{ij}^k v_k$. So decomposition numbers of representations decategorify to structure coefficients for multiplication, and these structure coefficients are guaranteed to be non-negative integers.

In the case of Λ, the category in question is the direct sum over the categories of complex representations of S_k for $k \geq 1$,

$$\mathcal{C} = \bigoplus_{n\geq 0} \mathrm{Rep}(\mathbb{C}[S_n]).$$

A representation V of S_k and a representation W of S_ℓ yield a representation $F_m(V, W) = \mathrm{Ind}_{S_k\times S_\ell}^{S_{k+\ell}}(V \otimes W)$; on K_0, this descends to the assignment $([V], [W]) \mapsto [\mathrm{Ind}_{S_k\times S_\ell}^{S_{k+\ell}}(V \otimes W)]$, which coincides with the usual multiplication. Restriction between the same subgroups descends to comultiplication. The special bases mostly have representation theoretic meaning: complete functions come from Ind-products of trivial representations, elementary functions come from Ind-products of sign representations, and Schur functions come from irreducible representations. The bialgebra axiom is a consequence of the Mackey Theorem on induction and restriction. The original description of this structure, though without the language of categorification, is due to Geissinger in the 1970s [Gei77]; see also [Zel81].

Since the bilinear form is a hom-space dimension,

$$\langle [V], [W]\rangle = \dim \mathrm{Hom}_{\mathcal{C}}(V, W),$$

it follows that the bilinear form will take values in $\mathbb{Z}_{\geq 0}$ when applied to elements equal to the classes of objects of the category. Homomorphisms are the higher expressions of bilinear form values. More generally, when $\mathcal{C}$ categorifies A, it is the morphisms of $\mathcal{C}$ that carry the new information.

The finite variable quotient of Λ, the ring of symmetric polynomials in n variables Λ_n, is Morita equivalent to the nilHecke algebra NH_n on n strands. This algebra was first introduced by Kostant and Kumar in the 1980s in the context of geometric representation theory [KK86]. Much more recently, Khovanov and Lauda introduced certain graded algebras in the categorification of quantum groups; in the simplest case of their construction, the nilHecke algebras categorify (an integral divided powers form of) $U_q^+(\mathfrak{sl}_2)$.

Thus the symmetric functions fit into categorification in two different ways, as the following figure illustrates:

$$\begin{tikzcd}
 & & S_k \arrow[d, rightsquigarrow] \\
\mathrm{NH}_n \arrow[r, dash, dashed] \arrow[d, rightsquigarrow] & \Lambda_n & \Lambda \arrow[l, twoheadrightarrow] \\
U_q^+(\mathfrak{sl}_2) & &
\end{tikzcd}$$

Squiggly lines denote decategorification (some sort of K_0) and dashed lines represent Morita equivalence.

To relate this to topology: the quantum group $U_q(\mathfrak{sl}_2)$ can be used to give a construction of the Jones polynomial, a Laurent polynomial invariant of links [Tur88; RT90]. The ingredients in the Reshetikhin-Turaev recipe are:

1. a quantum group $U_q(\mathfrak{g})$ and an integrable highest weight λ;

2. tensor products of the simple modules V_λ and V_λ^*;

3. various intertwiners between $\mathbb{C}(q)$ and certain tensor products of V_λ and V_λ^* (part of the braided monoidal category data).

A closed link is sent to a linear map $\mathbb{C}(q) \to \mathbb{C}(q)$ which turns out to be (up to normalization) a Laurent polynomial.

In the 1990s, Khovanov categorified the Jones polynomial—that is, he used a TQFT to construct a bigraded abelian group called *Khovanov homology* whose graded Euler characteristic is the Jones polynomial. One is led to ask: can the entire representation-theoretic approach of Reshetikhin-Turaev be categorified, rather than just the reuslt?

Webster answered this question in the affirmative by, building on the work of Khovanov-Lauda [KL09; KL11; KL10] and Rouquier [Rou08] on the categorification of quantum groups, categorifying tensor products of simple highest weight representations [Web10a] and the intertwiners between these representations [Web10b]. This gives a construction of link homology groups for any highest weight data $(\mathfrak{g}, \lambda)$; the case of $\mathfrak{g} = \mathfrak{sl}_2$, $\lambda = \frac{1}{2}\alpha$ (the fundamental weight) recovers Khovanov homology. In this setting the ingredients for the categorified recipe are:

1. a 2-category $\dot{\mathcal{U}}_q(\mathfrak{g})$ that categorifies $U_q(\mathfrak{g})$ [Lau10; KL09; KL10];
2. categorifications $\mathcal{T}^{\underline{\lambda}}$ of tensor products of tensor products of simples [Web10a];
3. functors between the derived categories $D^+(\mathcal{T}^{\underline{\lambda}})$ which categorify the intertwiners [Web10b].

A closed link is sent to an endofunctor of the derived category of $\mathbb{Z}$-graded $\Bbbk$-vector spaces; the homology of the image of $\Bbbk$ is then a bigraded abelian group which categorifies the Reshetikhin-Turaev link invariant associated to $(\mathfrak{g}, \lambda)$.

Link homology is not unique. In 2007, Ozsváth, Rasmussen, and Szabó used a projective TQFT to construct *odd Khovanov homology* [ORS07], a categorification of the Jones polynomial which is isomorphic to the usual ("even") Khovanov homology over $\mathbb{Q}$ but not over $\mathbb{Z}$ or $\mathbb{Z}/2$. Even and odd Khovanov homology each can distinguish links that the other cannot. The difference between the two constructions can be summed up by the motto: "replace symmetric products with exterior products."

The question which motivated the work of this thesis is the following: *Can the 2-representation theoretic approach of Webster be used to construct odd Khovanov homology from a categorification of $U_q^+(\mathfrak{sl}_2)$ that is inequivalent to the category of modules for the nilHecke algebra?*

The first major step in this direction, and the main theorem of this thesis, is the following.

Theorem. There is a family of superalgebras whose categories of supermodules categorify the integral divided powers form of $U_q^+(\mathfrak{sl}_2)$ as a q-bialgebra.

Chapters 3 and 4 prove this theorem and carry out a detailed study of these *odd nilHecke algebras.*

The center of the nilHecke algebra NH_n is a copy of the symmetric polynomials Λ_n. The natural analogue of this in the odd setting is not to take the center, but instead to take the basic algebra Morita equivalent to NH_n; this is a subalgebra $O\Lambda_n$ of a skew polynomial ring, and it plays the role of symmetric polynomials in the odd setting. And just as Λ can be realized as the limit of the system of the algebras Λ_n, one can take the limit of the superalgebras $O\Lambda_n$. The result, denoted $O\Lambda$, is the Hopf superalgebra of *odd symmetric functions.*

This object has turned out to be quite interesting in its own right.

Theorem. There is a Hopf superalgebra $O\Lambda$ that exhibits signed analogues of many of the familiar features of the symmetric functions: combinatorial bases with integral structure coefficients, Schur orthonormality, Littlewood-Richardson coefficients, and so forth. Modulo 2, these constructions agree with the corresponding even case constructions.

This superalgebra is studied in Chapters 1 and 2. The study of Λ and $O\Lambda$ fits naturally into a story depending on a deformation parameter $q \in \Bbbk^\times$; Λ corresponds to $q = 1$ and $O\Lambda$ corresponds to $q = -1$.

The present state of affairs, then, is summarized by the diagram:

$$\begin{tikzcd}
 & & ? \arrow[d, rightsquigarrow] \\
\mathrm{ONH}_n \arrow[r, dashed, dash] \arrow[d, rightsquigarrow] & O\Lambda_n & O\Lambda \arrow[l, twoheadrightarrow] \\
U_q^+(\mathfrak{sl}_2) & &
\end{tikzcd}$$

This thesis is devoted to the study of $O\Lambda$, ONH_n, their interrelationship, and odd categorification. The analogue of S_k in the diagram above (the "?") and the exploration of the story at other values of q are left to future study.

Outline

Chapter 1 contains the background necessary for the results in this thesis. Section 1.1 discusses how algebra is done in a braided monoidal category. We will need this general setting at first, but then spend most of our time doing superalgebra, which is a very special case. In Sections 1.2 and 1.3, the classical structures we seek to "oddify" are reviewed: symmetric functions and the 2-representation theory of $U_q^+(\mathfrak{sl}_2)$.

Having established the preliminaries, Chapter 2 studies the combinatorial algebras behind odd categorification. Section 2.1 reviews the q-Hopf algebra of quantum noncommutative symmetric functions $N\Lambda^q$, a deformation of the noncommutative symmetric functions. A certain quotient $N\Lambda^q$ gives a q-analogue of the symmetric functions, Λ^q. At $q = 1$, this is the usual algebra of symmetric functions. Section 2.2 studies this algebra at $q = -1$, in which case we call this the *odd symmetric functions*, $O\Lambda$. Odd analogues of the basic structions on Λ are discussed. The chapter closes with Section 2.3, which proposes more structures to find odd analogues of as well as a context within which to situate $O\Lambda$.

Chapter 3 introduces the skew polynomial model for $O\Lambda_n$ (Sections 3.1, 3.3) and the odd nilHecke algebra (Sections 3.2, 3.6). At this point diagrammatic algebra is introduced (Sections 3.4, 3.5) as a notational tool; as the expressions involved become more complicated, diagrammatics become more necessary for comprehension. Finally, Section 3.7 proves that the various definitions of odd Schur function all agree and establishes an odd Littlewood-Richardson rule.

The last chapter, Chapter 4, brings us to categorification. Section 4.1 introduces a "thick calculus" for the odd nilHecke algebra: this is a diagrammatic way of studying indecomposable projective modules. Section 4.2 introduces the cyclotomic quotients of ONH_n, which categorify simple $U_q(\mathfrak{sl}_2)$-modules, and Section 4.3 proves our main categorification theorem: that the odd nilHecke algebras categorify the divided power form of $U_q^+(\mathfrak{sl}_2)$. The chapter closes with Section 4.4, which summarizes progress on odd categorification since the results of this thesis were first proven.

An appendix contains some data: the bilinear form in the complete basis for unspecified q and for $q = -1$ as well expressions for odd elementary, power sum, and Schur functions in the complete basis.

The description of future work can be found in Sections 2.3 and 4.4.

The work described in this thesis is primarily taken from the papers [EK12; EKL12; Ell12], some of it verbatim.

Notation

$[n]$	balanced q-number	(1.3.1)
$[n]!$	balanced q-factorial	(1.3.1)
$\begin{bmatrix} n \\ k \end{bmatrix}$	balanced q-binomial coefficient	(1.3.1)
$\binom{\alpha}{k}$, for α a composition	the summation $\sum_j \binom{\alpha_j}{k}$	
$\lvert\alpha\rvert$, for α a composition	the summation $\sum_j \alpha_j$	
$SSYT(\lambda)$	semistandard Young tableaux of shape λ	Subsection 1.2.1
$SSYT(\lambda, \mu)$	semistandard Young tableaux of shape λ, content μ	Subsection 1.2.1
$w_r(T)$	row word of a tableau T	Subsection 2.2.4
α^{rev}	reverse of a composition α	
$(A)_n$	degree n part of a graded algebra A	
$M\{k\}$	module obtained by shifting gradings up by k	
$\lvert x\rvert, \deg(x)$	degree of x in a graded algebra	
Λ	symmetric functions	Subsection 1.2.1
Λ_n	symmetric polynomials in n variables	Subsection 1.2.1
$O\Lambda$	odd symmetric functions	Subsection 2.2.1
$O\Lambda_n$	odd symmetric polynomials in n variables	Subsection 3.1.1

Chapter 1

Preliminaries

This chapter reviews the necessary background for the later chapters. The brief Section 1.1 sets up conventions for studying Hopf algebras in braided categories, which is a natural setting for algebras that admit graded categorifications.

The two subsequent sections each review results that we will generalize later; essentially no proofs are given here. Section 1.2 reviews the Hopf algebra structure on symmetric functions Λ, the relationship to symmetric groups, and certain bases of Λ. Next, in Section 1.3, the diagrammatic categorification of $U_q(\mathfrak{sl}_2)$ and its simple representations is reviewed.

None of the results in this chapter are those of the author.

1.1 Braided algebra

Fix a commutative base ring $\Bbbk$ and a unit $q \in \Bbbk$. Let $\mathcal{M}od(\Bbbk)_q^{gr}$ be the category

$$\mathcal{M}od(\Bbbk)_q^{gr} \begin{cases} \text{objects:} & \mathbb{Z}\text{-graded } \Bbbk\text{-modules} \\ \text{morphisms:} & \text{degree preserving } \Bbbk\text{-module homomorphisms.} \end{cases}$$

If V is an object and $v \in V$ is homogeneous, write $|v|$ for the degree of v. We equip $\mathcal{M}od(\Bbbk)_q^{gr}$ with the monoidal structure given by tensor product of $\Bbbk$-modules and the braiding defined on homogeneous elements by

$$\begin{aligned} \tau_q : V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto q^{|v||w|} w \otimes v. \end{aligned} \tag{1.1.1}$$

Sometimes we write $\tau_q(V, W)$ to emphasize the objects in question. The category $\mathcal{M}od(\Bbbk)_q^{gr}$ is symmetric monoidal if and only if $q^2 = 1$ in $\Bbbk$. The examples most relevant to this thesis are:

- $\Bbbk = \mathbb{Z}$ or a field, $q = 1$: ordinary algebra
- $\Bbbk = \mathbb{Z}$ or a field, $q = -1$: superalgebra
- $\Bbbk = \mathbb{Z}[q]/(\Phi_n(q))$, where $\Phi_n(q)$ is the n-th cyclotomic polynomial (or take $\Bbbk = \mathbb{C}$ and q to be a primitive n-th root of unity): "anyonic" algebra, so called by physicists
- $\Bbbk = \mathbb{Z}[q, q^{-1}]$ or $F[q, q^{-1}]$ for a field F (or take $\Bbbk = \mathbb{C}$ and q generic)

Bialgebra and Hopf algebra objects in the braided monoidal category $\mathcal{M}od(\Bbbk)_q^{gr}$ are called "q-bialgebras," "q-Hopf algebras" in the combinatorial literature [AM11] and "twisted bialgebras," "twisted Hopf algebras" in the representation theoretic literature [Lus93], [KL09]. Since the word "twisted" is quite overloaded in mathematics, we will adopt the combinatorialists' terminology.

It is worth taking the time to be precise about what is meant by a q-bialgebra. Viewing an algebra as generators and relations on a $\Bbbk$-module, the braiding makes no difference whatsoever. The difference is in the canonical algebra structure given to tensor products: if A, B are q-algebras (i.e., algebra objects in $\mathcal{M}od(\Bbbk)_q^{gr}$), then $A \otimes B$ is a q-algebra with multiplication

$$m_{A\otimes B} = (m_A \otimes m_B)(\mathbb{1}_A \otimes \tau_q(B, A) \otimes \mathbb{1}_B), \tag{1.1.2}$$

as maps $A \otimes B \otimes A \otimes B \to A \otimes B$. On elements, this is

$$(a \otimes 1)(1 \otimes b) = a \otimes b,$$
$$(1 \otimes b)(a \otimes 1) = q^{|a||b|} a \otimes b$$

for homogeneous $a \in A$, $b \in B$. By identifying A with $A \otimes 1$ and B with $1 \otimes B$, the previous equation is usually written as

$$ab = q^{|a||b|} ba$$

and called the "Koszul rule of signs." So for an algebra (A, m, η) and a coalgebra structure (A, Δ, ε) on the same underlying $\Bbbk$-module, $(A, m, \Delta, \eta, \varepsilon)$ is a q-bialgebra if and only if

$$\Delta m = (m \otimes m)(\mathbb{1} \otimes \tau_q \otimes \mathbb{1})(\Delta \otimes \Delta) \tag{1.1.3}$$

(plus the analogous requirement for η, ε). Equation (1.1.3) is equivalent to the statement that $\Delta : A \to A \otimes A$ is a homomorphism of algebras, where $A \otimes A$ is given the multiplication (1.1.2).

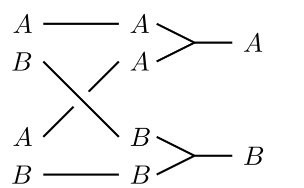
Example 1.1.1. The basic example of a q-bialgebra is the algebra of q-commuting polynomials,

$$\mathrm{Pol}_n^q = \Bbbk\langle x_1, \ldots, x_n\rangle / \langle x_j x_i = q x_i x_j \text{ if } i < j\rangle,$$

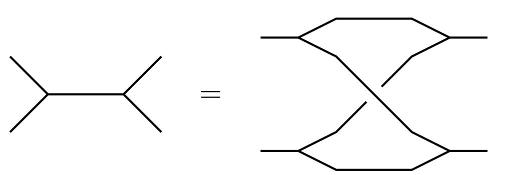
equipped with the comultiplication determined by $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$. For $q = 1$ this is the usual commutative polynomial ring over $\Bbbk$, and for $q = -1$ this is the ring of skew polynomials over $\Bbbk$. For any q, there is the evident isomorphism

$$(\mathrm{Pol}_1^q)^{\otimes n} \cong \mathrm{Pol}_n^q.$$

Remark 1.1.2. The definition of $m_{A\otimes B}$ can be summarized by the diagram



This picture is read left to right. Strands represent objects, merges represent multiplication, and crossings represent the braiding τ_q. Using splits to represent comultiplication, the q-bialgeba axiom takes the diagrammatic form


(1.1.4)

1.2 Symmetric functions and categorification

This section reviews the representation theoretic interpretation of certain structures on the ring of symmetric functions (in infinitely many variables, so we will be dealing with the symmetric group rather than GL_n). As in the previous section, $\Bbbk$ is a commutative ring. The standard references for the results in this section are [Sta99], [Ful97], [Gei77].

1.2.1 Symmetric polynomials and symmetric functions

Let $\mathrm{Pol}_n = \Bbbk[x_1, \ldots, x_n]$, considered as a graded algebra with $|x_i| = 1$ for each i. The group S_n acts on Pol_n by $w \cdot x_i = x_{w(i)}$. Define the *elementary symmetric polynomials*

$$e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} \cdots x_{i_n} \tag{1.2.1}$$

and the *complete symmetric polynomials*

$$h_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} x_{i_1} \cdots x_{i_n}. \tag{1.2.2}$$

The "Fundamental Theorem of Symmetric Polynomials" states that the S_n-invariants of Pol_n are described by

$$(\mathrm{Pol}_n)^{S_n} = \Bbbk[e_1, \ldots, e_n] = \Bbbk[h_1, \ldots, h_n].$$

This algebra of invariants is the ring of symmetric polynomials in n variables, denoted Λ_n. The maps

$$\mathrm{Pol}_{n+1} \to \mathrm{Pol}_n$$
$$x_{n+1} \mapsto 0, \qquad x_i \mapsto x_i \text{ for } 1 \leq i \leq n,$$

take Λ_{n+1} to Λ_n and send $e_k \mapsto e_k$, $h_k \mapsto h_k$. The limit of the resulting system,

$$\Lambda = \varprojlim \Lambda_n,$$

taken in the category of graded $\Bbbk$-algebras, is the algebra of *symmetric functions*. Equivalently, Λ is the algebra of all S_n-invariant power series in $x_1, x_2, \ldots$ of bounded degree. This limit is generated as a polynomial algebra by either the e_k's or the h_k's,

$$\Lambda = \Bbbk[e_1, e_2, \ldots] = \Bbbk[h_1, h_2, \ldots]. \tag{1.2.3}$$

By convention we write $e_0 = 1$ and $e_k = 0$ when $k < 0$. A consequence of the "Fundamental Theorem" is that

$$\{e_\lambda\}_{\lambda \vdash k, \ell(\lambda) \leq n}, \{h_\lambda\}_{\lambda \vdash k, \ell(\lambda) \leq n} \text{ are both bases for } \Lambda_n \text{ in degree } k,$$
$$\{e_\lambda\}_{\lambda \vdash k}, \{h_\lambda\}_{\lambda \vdash k} \text{ are both bases for } \Lambda \text{ in degree } k,$$

where "$\lambda \vdash k$" means λ is a partition of k and $\ell(\lambda)$ is the number of nonzero parts of λ. We call these the *elementary* and *complete* bases. Another useful notation is "$\alpha \vDash k$," which means that α is a composition of k.

Some combinatorial notions and notation we will make use of: a Young tableau of shape λ with respect to some ordered alphabet A is an assignment of a letter from A to each box of some Young diagram. The shape $\mathrm{sh}(T)$ of a tableau T is the underlying Young diagram and the content $\mathrm{ct}(T)$ is the composition α such that α_i is the number of i's in T. A tableau is *semistandard* if its entries strictly increase in columns and weakly increase in rows. We write $SSYT(\lambda)$ for the set of semistandard Young tableaux of shape λ and $SSYT(\lambda, \mu)$ for the subset of those which have content μ.

There are several other bases of Λ which are interesting; we describe three. The *monomial function* m_λ is the sum of all monomials "of shape λ":

$$m_\lambda = \sum_\alpha x^\alpha, \tag{1.2.4}$$

where $\lambda = (\lambda_1, \ldots, \lambda_r, 0, \ldots)$ is a partition padded with infinitely many zeroes at the end and the sum ranges over all *distinct* permutations α of λ, and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. The monomial functions form an integral basis of Λ.

For each $k \geq 1$, the k-th power sum function is

$$p_k = \sum_j x_j^k$$

and we define $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_r}$. Then the products p_λ form a basis over any field of characteristic zero.

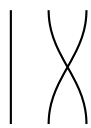
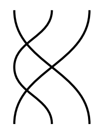
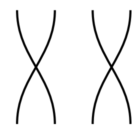
The most interesting is the basis of Schur functions. One definition of the Schur function s_λ is

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathrm{ct}(T)},$$

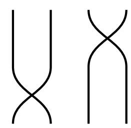
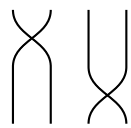
where $SSYT(\lambda)$ is the set of all semistandard Young tableaux of shape λ. The Schur functions form a fascinating integral basis. We will see them repeatedly in what follows.

1.2.2 Symmetric group representations

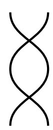
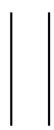
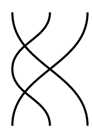
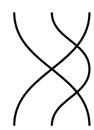
We briefly review the complex representation theory of the symmetric groups S_n, $n \geq 1$. Elements of S_n can conveniently be drawn as "strands diagrams," e.g. (using one-line notation for permutations)

(132) =  , (321) =  , (2143) =  .

The only diagrammatic relations are isotopy, which includes the relation

 =  , (1.2.5)

and the diagrammatic Coxeter relations

 =  ,  =  . (1.2.6)

To be more precise: let

$$s_i = (i \quad i+1)$$

be the i-th simple transposition (in S_n, for $1 \leq i \leq n-1$) and fix once and for all a reduced Coxeter expression $s_{i_1} \cdots s_{i_r}$ for each element $w \in S_n$. Any word $w = s_{j_1} \cdots s_{j_r}$ can be drawn as a strands diagram: reading the word left to right, draw crossings from bottom to top, where s_j is drawn as a crossing between the j-th and $(j+1)$-st strands. Then the relations (1.2.5),(1.2.6) suffice to make any strands diagram equal one of the chosen reduced expression diagrams.

We now construct the irreducible representations of S_n, following an elegant diagrammatic argument the author learned from Mikhail Khovanov. For any composition α of n (we write $\alpha \vDash n$), let $S_\alpha = S_{\alpha_1} \times \cdots \times S_{\alpha_r}$ be the corresponding Young subgroup. Write $\underline{\mathbb{C}}, \underline{\mathbb{C}}^-$ for the trivial and sign representations of S_n respectively and define

$$I_\alpha = \mathrm{Ind}_{S_\alpha}^{S_n}(\underline{\mathbb{C}}), \qquad I_\alpha^- = \mathrm{Ind}_{S_\alpha}^{S_n}(\underline{\mathbb{C}}^-).$$

In general, if $H \subset G$ is a subgroup of a finite group G and $\chi : G \to \mathbb{C}^\times$ is a character, then

$$e_{H,\chi} = \frac{1}{|H|} \sum_{h \in H} \chi(h) h$$

is an idempotent, and

$$\mathbb{C}[G]e_{H,\chi} \cong \mathrm{Ind}_H^G(\underline{\mathbb{C}}_\chi) \tag{1.2.7}$$

as G-representations, where $\underline{\mathbb{C}}_\chi$ is the one dimensional representation of character χ. Even more generally, if e, e' are idempotents in an algebra A, then

$$Hom_A(Ae, Ae') = eAe'. \tag{1.2.8}$$

Combining equations (1.2.7) and (1.2.8) and using the diagrammatic presentation of $\mathbb{C}[S_n]$, it follows that we can compute the hom-space from $I_\mu^\pm$ to $I_\lambda^\pm$ simply by counting linearly independent diagrams in the space

$$\left(\sum_{g\in S_\mu}(\pm 1)^{\ell(g)}g\right)\cdot\mathbb{C}[S_n]\cdot\left(\sum_{g\in S_\lambda}(\pm 1)^{\ell(g)}g\right).$$

Proposition 1.2.1. Let λ, μ be partitions of n.

1. If $\mu > \lambda^T$ (lexicographic order), then $\mathrm{Hom}_{S_n}(I_\mu, I^-_{\lambda^T}) = 0$.

2. The space $\mathrm{Hom}_{S_n}(I_\lambda, I^-_{\lambda^T})$ is one dimensional.

We sketch the proof of this Proposition for two reasons. Firstly, the argument is an example of the elegance and cleanliness of diagrammatics. Secondly, it motivates the construction of Subsection 2.1.2.

Sketch of proof. We first reduce to a purely combinatorial problem. By (1.2.8), the dimension of a hom-space between representations that can be picked out of $\mathbb{C}[S_n]$ by idempotents e, e' equals the number of linearly independent elements of $e\mathbb{C}[S_n]e'$. The idempotent defining I_α is

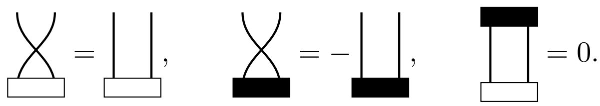
$$e_\alpha^+ = \frac{1}{|S_\alpha|}\sum_{g\in S_\alpha} g$$

and the idempotent defining I_α^- is

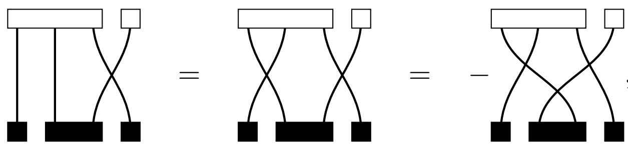
$$e_\alpha^- = \frac{1}{|S_\alpha|}\sum_{g\in S_\alpha}(-1)^{\ell(g)}g.$$

Diagrammatically, in computing $\mathrm{Hom}_{S_n}(I_\mu, I^-_{\lambda^T})$, we symmetrize over the first μ_1 strands on top, the next μ_2 strands on top, and so forth; then we anti-symmetrize over the first

λ_1^T strands on bottom, the next λ_2^T strands on bottom, and so forth. If we use a white platform grouping several strands to denote symmetrization and a black platform to denote anti-symmetrization, then we have the diagrammatic rules



These rules also hold if more strands are present on the given platforms. For example, viewing diagrams as intertwiners $\mathbb{C}[S_n]e^+_{31} \to \mathbb{C}[S_n]e^-_{121}$,



and so forth. The two simplificiations we gain are:

- any diagram in which a white and a black platform are connected by more than one strand is zero;
- and two diagrams which differ by a crossing that can be absorbed into a platform (of either color) are linearly dependent; thus consider only the one in which all redundant crossings are absorted.

The rest of the argument follows the proof of Proposition 2.2.5 below, where such simplified diagrams are called "lite" diagrams. $\square$

Corollary 1.2.2. There is a unique irreducible representation which appears in the decomposition of both I_λ and $I^-_{\lambda^T}$; call it L_λ. Then $L_\lambda \cong L_\mu$ if and only if $\lambda = \mu$, and the set $\{L_\lambda\}_{\lambda \vdash n}$ is a complete set of (isomorphism class representatives of) irreducible representations of S_n.

Proof. Induct downward in the dominance partial order, using Proposition 1.2.1 and the semi-simplicity of the category of representations of S_n: a one-dimensional hom-space between V and W is equivalent to V and W sharing exactly one irreducible representation, and with multiplicity 1. $\square$

1.2.3 Grothendieck groups

Various sorts of Grothendieck group are used in categorification, but for our purposes, two closely related ones will do. For any essentially small category (that is, its isomorphism classes of objects form a set and not a proper class) $\mathcal{C}$, let $\underline{\mathcal{C}}$ denote the free abelian group on its set of isomorphism classes of objects; we write $[X]$ for the image of the object X in $\underline{\mathcal{C}}$.

If $\mathcal{C}$ is an additive category, its *split Grothendieck group* is defiend to be

$$K_0(\mathcal{C}) = \underline{\mathcal{C}}/([B] = [A] + [C] \text{ if } B \cong A \oplus C).$$

If $\mathcal{A}$ is an abelian category, its *Grothendieck group* (no modifying adjective) is defined to be

$$G_0(\mathcal{C}) = \underline{\mathcal{C}}/([B] = [A] + [C] \text{ if there is a short exact sequence } 0 \to A \to B \to C).$$

For a ring R, let $\mathcal{M}od(R)$ be its category of finitely generated (say, left) modules and $\mathcal{P}roj(R)$ be its category of finitely generated projective modules. Define

$$G_0(R) = G_0(\mathcal{M}od(R)), \qquad K_0(R) = K_0(\mathcal{P}roj(R)).$$

There are various ways of getting a product structure on Grothendieck groups. The easiest is the following: An exact functor $F : \mathcal{C} \to \mathcal{D}$ induced a homomorphism of Grothendieck groups $K_0(F) : K_0(\mathcal{C}) \to K_0(\mathcal{D})$. In particular, if R is a ring and M is an (R, R)-bimodule which is two-sided projective, then the tensor product with M induces an endomorphism of $K_0(R)$; for R commutative, then, $K_0(R)$ is a ring with multiplication coming from the tensor product of modules.

However, we will use a slightly different construction of multiplication. Suppose we have a nested family of algebras

$$\cdots \subseteq A_k \subseteq A_{k+1} \subseteq A_{k+1} \subseteq \cdots$$

as well as inclusions

$$A_k \otimes A_\ell \hookrightarrow A_{k+\ell}.$$

This situation is sometimes called a "tower of algebras." (More general additive posets can also be used in place of linear inclusions, e.g. the positive integral weights of a Kac-Moody algebra.) Given an A_k-module V and an A_ℓ module W, the tensor product $V \otimes W$ is an $A_k \otimes A_\ell$ module and we can form the induced module

$$\mathrm{Ind}_{k,\ell}^{k+\ell}(V \otimes W) = A_{k+\ell} \otimes_{A_k \otimes A_\ell} (V \otimes W).$$

If the tower is such that this induced module is projective whenever V and W are, then we have a bilinear map

$$K_0(A_k) \otimes K_0(A_\ell) \to K_0(A_{k+\ell}).$$

In this situation, the direct sum of these Grothendieck groups

$$K_0(A_\bullet) = \bigoplus_{k \in \mathbb{Z}} K_0(A_k) \tag{1.2.9}$$

becomes a graded ring, an element $[V]$ corresponding to an A_k-module V is homogeneous of degree k. Examples of this construction abound: symmetric groups [Gei77], finite classical groups and wreath products [Zel81], nilHecke and other Khovanov-Lauda-Rouquier algebras [KL09], 0-Hecke algebras [TU96]; there are many more.

One notion of categorification, the one which we use in this thesis, is the act of replacing an algebra acting on a module by a category acting on another category in such a way that applying K_0 to the latter recovers the former. Our first example, which we will describe in the following subsection, is well known and rather old—the complex representations of S_n categorify the Hopf algebra of symmetric functions.

1.2.4 Frobenius and Geissinger

The classic example of the tower of algebras construction from the previous subsection is the categorification of Λ by the algebras $\mathbb{C}[S_n]$ (which plays the role of A_n in the notation above). For this section, we take $\Bbbk = \mathbb{Q}$ and, for any algebra A, set $K_0(A)_\mathbb{Q} = K_0(A) \otimes_\mathbb{Z} \mathbb{Q}$. By Corollary 1.2.2, $K_0(\mathbb{C}[S_n])_\mathbb{Q}$ is isomorphic (as vector spaces) to the degree n part of Λ. Taking the direct sum over all $n \geq 0$, we have two highly structured objects.

	symmetric groups	symmetric functions
the object	$K_0(\mathbb{C}[S_\bullet])_\mathbb{Q}$	Λ
$\mathbb{Z}$-grading	degree n part: $K_0(\mathbb{C}[S_n])_\mathbb{Q}$	$\deg(h_n) = n$
multiplication	$[V] \cdot [W] = \mathrm{Ind}^n_{k,\ell}(V \otimes W)$	$h_\alpha \cdot h_\beta = h_{\alpha\beta}$
comultiplication	$\Delta([V]) = \sum_{k+\ell=n}[\mathrm{Res}^{k+\ell}_{k,\ell}(V)]$	$\Delta(h_n) = \sum_{k=0}^n h_k \otimes h_n$
bilinear form	$([V],[W]) = \dim \mathrm{Hom}_{S_n}(V,W)$	$(h_\alpha, h_\beta) = \#(S_\alpha \backslash S_n / S_\beta)$

The beautiful result of Geissigner [Gei77] is that the *Frobenius characteristic map* is an isomorphism between these two graded Hopf algebras (so that we can think of the combinatorial structure as coming from representation theory).

For a partition μ, let m_i be the number of rows in μ of length equal to i. Then set

$$z_\mu = \frac{n!}{\#\{w \text{ of shape } \mu\}} = \prod_i i^{m_i} i!.$$

Recall that $K_0(\mathbb{C}[S_n])_\mathbb{Q}$ can be identified with the space of $\mathbb{Q}$-valued class functions on S_n. Under this identification, the degree n part of the Frobenius characteristic map is

$$\begin{aligned} \mathrm{ch}_n : K_0(\mathbb{C}[S_n])_\mathbb{Q} &\to \Lambda \\ f &\mapsto \frac{1}{n!}\sum_{w \in S_n} f(w) p_{\mathrm{shape}(w)} = \sum_{\mu \vdash n} z_\mu^{-1} f(\mu) p_\mu, \end{aligned} \tag{1.2.10}$$

where shape(w) is the partition representing the lenghts of the cycles when w is written in disjoint cycle notation.

Theorem 1.2.3 ([Gei77])**.** The Frobenius characteristic map

$$\mathrm{ch} : K_0(\mathbb{C}[S_\bullet])_\mathbb{Q} \to \Lambda_\mathbb{Q}$$

is an isomorphism of Hopf algebras.

Using this theorem, we can extend the table above to describe the representation theoretic meaning of certain bases of Λ. For a conjugacy class $C_\lambda \subseteq S_n$, let $\mathbf{1}_{C_\lambda}$ be the class function which is 1 on C_λ and 0 otherwise.

	representation of/class function on S_n	symmetric function
complete basis	$\mathrm{Ind}_{S_\lambda}^{S_n}(\underline{\mathbb{C}})$	h_λ
elementary basis	$\mathrm{Ind}_{S_\lambda}^{S_n}(\underline{\mathbb{C}}^-)$	e_λ
power sum basis	$z_\mu \mathbf{1}_{C_\lambda}$	p_λ
Schur basis	L_λ (irreducible)	s_λ

Let $\Lambda_k = \Lambda/(e_m : m > k)$ be the finite variable quotient, isomorphic to $\Bbbk[x_1, \dots, x_k]^{S_k}$. This is isomorphic to the character ring for polynomial representations of $GL_k(\mathbb{C})$. A Schur function s_λ is sent to 0 in this quotient if and only if the first column of λ has height greater than k, that is, if and only if $\lambda_1^T > k$. The subsequent quotient by complete functions is isomorphic to the singular cohomology ring of a complex Grassmannian

$$\Lambda_k/(h_m : m > n-k) \cong H^*(\mathrm{Gr}_k(\mathbb{C}^n), \Bbbk)$$

under the identification of e_i with the i-th Chern class of the tautological bundle τ and $(-1)^i h_i$ with the i-th Chern class of a complement to τ. In this quotient, s_λ goes to 0 if and only if either $\lambda_1 > n-k$ or $\lambda_1^T > k$. A nonzero Schur function s_λ in $H^*(\mathrm{Gr}_k(\mathbb{C}^n), \Bbbk)$ is the class representing a Schubert subvariety, which has a description in terms of linear enumerative geometry. This Grassmannian quotient will play a role in cyclotomic quotients of the nilHecke algebra below.

1.3 Categorified quantum groups: the $\mathfrak{sl}_2$ case

In this section we will review another tower of algebras construction: the (divided powers form of the) positive part of the quantum group $U_q(\mathfrak{sl}_2)$ is categorified by nilHecke algebras. This is the simplest nontrivial case of Khovanov-Lauda's categorification theorem [KL09; KL11; KL10]; see also [Rou08].

1.3.1 Quantum $\mathfrak{sl}_2$

The balanced q-number, q-factorial, and q-binomial coefficient are defined by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]! = \prod_{k=1}^{n} [k], \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}. \tag{1.3.1}$$

Definition 1.3.1. The *quantum group* (or, more correctly, *quantized enveloping algebra*) $U_q(\mathfrak{sl}_2)$ is the $\mathbb{C}(q)$-algebra defined by generators E, F, K, K^{-1} and relations

$$KK^{-1} = 1 = K^{-1}K \tag{1.3.2}$$

$$KE = q^2 EK, \qquad KF = q^{-2} FK, \tag{1.3.3}$$

$$EF - FE = \tfrac{K-K^{-1}}{q-q^{-1}}. \tag{1.3.4}$$

The algebra $U_q(\mathfrak{sl}_2)$ has a PBW decomposition (tensor products are over $\mathbb{C}(q)$, the isomorphism is of vector spaces),

$$U_q(\mathfrak{sl}_2) \cong U_q^+ \otimes U_q^0 \otimes U_q^-, \tag{1.3.5}$$

where U_q^+ is the subalgebra generated by E, U_q^0 is the subalgebra generated by K and K^{-1}, and U_q^- is the subalgebra generated by F.

Constructing Verma modules and simple modules for $U_q(\mathfrak{sl}_2)$ is completely analogous to the case of the universal enveloping algebra $U(\mathfrak{sl}_2)$, if one thinks of K as q^h. For $n \geq 0$, let M_n be the free U_q^--module generated by a cyclic vector v_0 and extend this to a representation of $U_q^- \otimes U_q^0$ by declaring $Kv_0 = q^n v_0$. Then the quotient

$$V_n = M_n/(M_n \cdot F^{n+1} v_0) \tag{1.3.6}$$

is a simple $U_q(\mathfrak{sl}_2)$-module of dimension $n+1$.

In categorification, the following version of $U_q^+(\mathfrak{sl}_2)$ is used instead of U_q^+ [KL09].

Definition 1.3.2 ([Lus93])**.** Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. The *integral divided powers form* of $U_q^+(\mathfrak{sl}_2)$, denoted $U_q^+(\mathbf{sl}_2)_{\mathcal{A}}$, is the free $\mathcal{A}$-module with basis $\{\theta^{(n)}\}_{n\geq 1}$ and multiplication

$$\theta^{(a)}\theta^{(b)} = \begin{bmatrix} a+b \\ a \end{bmatrix} \theta^{(a+b)}. \tag{1.3.7}$$

1.3.2 nilHecke algebras categorify $U_q^+(\mathfrak{sl}_2)$

In [KL09; KL11], Khovanov and Lauda gave a categorification of $U_q^+(\mathfrak{g})_{\mathcal{A}}$ for any symmetrizable Kac-Moody algebra $\mathfrak{g}$ using what are now called *KLR algebras*. In the $\mathfrak{g} = \mathfrak{sl}_2$ case of their construction, the KLR algebras are the nilHecke algebras NH_n of [KK86] (see also [Man01]). For $n \geq 1$, this algebra has generators $x_1, \ldots, x_n$ and $\partial_1, \ldots, \partial_{n-1}$ and relations

$$\begin{aligned}
&\partial_i^2 = 0, \quad \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}, \\
&x_i \partial_i - \partial_i x_{i+1} = 1, \quad \partial_i x_i - x_{i+1} \partial_i = 1, \\
&x_i x_j = x_j x_i \quad (i \neq j), \quad \partial_i \partial_j = \partial_j \partial_i \quad (|i-j| > 1), \\
&x_i \partial_j = \partial_j x_i \quad (i \neq j, j+1).
\end{aligned}$$

The algebra NH_n, as a module over itself, decomposes into $[n]!$ copies (graded direct sum) of the unique indecomposable projective P_n. From this fact, their main theorem in this simplest case follows:

Theorem ([KL09])**.** The nilHecke algebra NH_n categorify Lusztig's divided powers form $U_q^+(\mathbf{sl}_2)_{\mathcal{A}}$ as a q-bialgebra over $\mathbb{Q}(q)$: there is an isomorphism

$$K_0(\mathrm{NH}_\bullet) \otimes_{\mathbb{Z}} \mathbb{Q}(q) \cong \mathrm{NH}_n \otimes_{\mathbb{Z}} \mathbb{Q}(q)$$

sending the indecomposable project P_n to the generator $\theta^{(n)}$. Shifting the grading of a module becomes multiplication by q: $[M\{k\}] = q^k[M]$ in K_0.

Let $N \geq 0$ be a dominant integral weight for $\mathfrak{sl}_2$. The quotient

$$\mathrm{NH}_n^N = \mathrm{NH}_n / (x_1^N) \tag{1.3.8}$$

is called a *cyclotomic quotient* of NH_n. These algebras naturally carry NH_n-actions, and they categorify highest weight simples for $U_q^+(\mathfrak{sl}_2)_{\mathcal{A}}$:

Theorem ([LV11; KK11])**.** Let N be a dominant integral weight for $\mathfrak{sl}_2$ and let $V_{\mathcal{A}}^N$ be the highest weight simple of $U_q^+(\mathfrak{sl}_2)_{\mathcal{A}}$ of highest weight q^N. Then

$$K_0(\mathrm{NH}_\bullet^N) \cong V_{\mathcal{A}}^N$$

as representations of $U_q^+(\mathfrak{sl}_2)_{\mathcal{A}}$.

Remark 1.3.3. The nilHecke algebra and its cyclotomic quotients have interesting Morita equivalences:

$$\mathrm{NH}_n \cong \mathrm{End}_{\Lambda_n}(\mathrm{Pol}_n) \cong \mathrm{Mat}_{[n]!}(\Lambda_n), \qquad \mathrm{NH}_n^N \cong \mathrm{Mat}_{[n]!}(H^*(\mathrm{Gr}_n(\mathbb{C}^N)).$$

In their papers, Khovanov-Lauda give a diagrammatic definition of the KLR algebras. We will make heavy use of diagrammatic algebra in Chapters 3 and 4. Two excellent expositions of diagrammatic algebra are [Kho10] and [Lau11]; the reader unfamiliar with diagrammatic algebra is strongly encouraged to concult those sources rather than the extremely brief overview we give in Section 3.4.

Chapter 2

Symmetric functions, q-analogues, and variants

Throughout this chapter, we work in the category $\mathcal{M}od(\Bbbk)_q^{gr}$ defined in Section 1.1. So "algebra" means "algebra object in $\mathcal{M}od(\Bbbk)_q^{gr}$" and so forth. Most of the results in this chapter are from [EK12], which is joint work with Mikhail Khovanov.

If α is a composition and $k \geq 1$, we abbreviate

$$\binom{\alpha}{k} = \sum_j \binom{\alpha_j}{k}.$$

For a Young diagram λ and $X \in \{N, S, E, W, NE, NW, SE, SE\}$ (meaning "north," "south," and so forth), let $X(\lambda)$ be the sum

$$X(\lambda) = \sum_{\text{boxes } B \text{ in } \lambda} \#\{\text{boxes farther } X \text{ than } B\}.$$

If T is a tableau, $X(T) = X(\mathrm{sh}(T))$. If X is decorated with a comparison operator $* \in \{<, \leq, >, \geq\}$, then

$$X^*(\lambda) = \sum_{\text{boxes } B \text{ in } \lambda} \#\{\text{boxes farther } X \text{ than } B \text{ with entry } * \text{ that of } B\}.$$

Example 2.0.4.

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad N(T) = 15, \quad NE(T) = 7, \quad N^<(T) = 12, \quad NE^<(T) = 4.$$

2.1 Quantum noncommutative symmetric functions

2.1.1 Another construction of the symmetric functions

For this subsection only, set $q = 1$, so $\mathcal{M}od(\Bbbk)_q^{gr}$ is just the category of graded $\Bbbk$-modules. Let $N\Lambda$ be a free algebra on generators $h_1, h_2, \dots$ with $|h_k| = k$. Set $h_0 = 1$ and $h_k = 0$ for $k < 0$. The homogeneous part of $N\Lambda$ of degree k has a basis $\{h_\alpha\}_{\alpha \vDash k}$, where

$$h_\alpha = h_{\alpha_1} \cdots h_{\alpha_r} \text{ for a composition } \alpha = (\alpha_1, \dots, \alpha_r) \text{ of } k.$$

When the subscripts in use do not exceed 9, we will abbreviate further: $h_{223} = h_{(2,2,3)} = h_2^2 h_3$. This algebra $N\Lambda$ is just the ring (in fact, Hopf algebra) of *noncommutative symmetric functions* first introduced in [GKL+95]. Define a bilinar form on $N\Lambda$ by setting

$$(h_\alpha, h_\beta) = \#(S_\alpha \backslash S_k / S_\beta) \tag{2.1.1}$$

when $|\alpha| = |\beta| = k$ and 0 otherwise.

Proposition 2.1.1. The radical R of this bilinear form is the two-sided algebra ideal generated by all commutators $h_a h_b - h_b h_a$. Thus $N\Lambda/R \cong \Lambda$, with the image of h_α being h_λ for the unique partition λ which is a rearrangement of α.

The proof is easy; see an analogous but more complicated argument below, Proposition 2.2.2. In fact $N\Lambda$ admits a Hopf algebra structure and R is a Hopf ideal, so that this gives another construction of Λ as a Hopf algebra.

2.1.2 Deforming the diagrammatic construction

Throughout this subsection, the reader may wish to keep in mind the diagrammatic construction of the irreducible representations of the symmetric group from the previous chapter.

Now we return to the case of unspecified q and deform the above construction. Let $N\Lambda^q$ be a free algebra on generators $h_1, h_2, \dots$ with $|h_k| = k$. Set $h_0 = 1$ and $h_k = 0$ for $k < 0$. The homogeneous part of $N\Lambda^q$ of degree k has a basis $\{h_\alpha\}_{\alpha \vDash k}$, where

$$h_\alpha = h_{\alpha_1} \cdots h_{\alpha_r} \text{ for a composition } \alpha = (\alpha_1, \dots, \alpha_r) \text{ of } k.$$

Replace equation (2.1.1) by

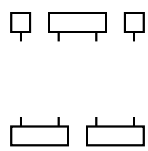
$$(h_\alpha, h_\beta)_q = \sum_{C \in S_\alpha \backslash S_k / S_\beta} q^{\ell(w_C)}, \tag{2.1.2}$$

where w_C is the minimal Coxeter length representative of the double coset C and ℓ is the Coxeter length function. Elements of unequal homogeneous degrees are orthogonal. Extend the bilinear form to $N\Lambda^q \otimes N\Lambda^q$ multiplicatively with no use of the braiding,

$$(w \otimes x, y \otimes z)_q = (w, y)_q (x, z)_q. \tag{2.1.3}$$

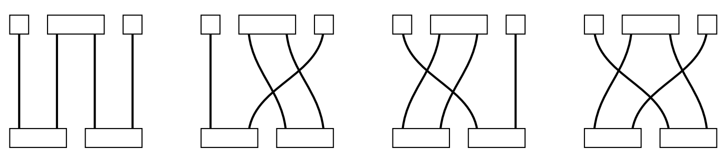
Computing products $(h_\alpha, h_\beta)_q$ can be done diagrammatically as follows, with the running example of $\alpha = (1, 2, 1)$, $\beta = (2, 2)$.

1. Among the bottom points of a strands diagram for S_k, group together into "platforms" the first β_1, then the next β_2, and so forth. Do the same for the top points with α in place of β.



2. From the $k!$ possible reduced expression diagrams, set $A(\alpha, \beta)$ be the set of those such that no two strands that begin or end at the same platform ever cross.

$A((1, 2, 1), (2, 2))$:



3. Define

$$(h_\alpha, h_\beta)_q = \sum_{D \in A(\alpha,\beta)} q^{\#\text{ of crossings in } D}. \tag{2.1.4}$$

In the example, $(h_{121}, h_{22})_q = 1 + 2q^2 + q^3$.

Define a coproduct on $N\Lambda^q$ by setting

$$\Delta(h_k) = \sum_{i=0}^{k} h_i \otimes h_k \tag{2.1.5}$$

and extending to $N\Lambda^q$ so as to make Δ a homomorphism of q-algebras. For example,

$$\begin{aligned}\Delta(h_{12}) &= (h_1 \otimes 1 + 1 \otimes h_1)(h_2 \otimes_1 + h_1 \otimes h_1 + 1 \otimes h_2)\\ &= h_{12} \otimes 1 + h_{11} \otimes h_1 + h_1 \otimes h_2 + q^2 h_2 \otimes h_1 + q h_1 \otimes h_{11} + 1 \otimes h_{12}.\end{aligned}$$

The counit on $N\Lambda^q$ sends all positive degree elements to zero,

$$\varepsilon(x) = 0 \text{ if } x \text{ is homogeneous of positive degree.} \tag{2.1.6}$$

The algebra $N\Lambda^q$ now has the structure of a q-bialgebra; later we will describe its antipode, making it a q-Hopf algebra. This q-Hopf algebra is non-q-cocommutative and is free associative as an algebra.

Define elements $e_k \in N\Lambda^q$ by $e_k = 0$ for $k < 0$, $e_0 = 1$, and

$$\sum_{i=0}^{k} (-1)^i q^{\binom{i}{2}} e_i h_{k-i} = 0 \text{ for } k \geq 1. \tag{2.1.7}$$

Or, equivalently,

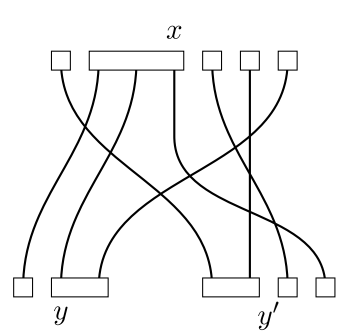
$$e_n = q^{-\binom{n}{2}} \sum_{\alpha \vDash n} (-1)^{\ell(\alpha)-n} h_\alpha. \tag{2.1.8}$$

Proposition 2.1.2. Multiplication and comultiplication in $N\Lambda^q$ are adjoint with respect to $(\cdot,\cdot)_q$. That is,

$$(x, yy')_q = (\Delta(x), y \otimes y')_q \tag{2.1.9}$$

for all $x, y, y' \in N\Lambda^q$.

Proof. It suffices to prove (2.1.9) when x, y, y' are products of h_k's. In this case, there is an apparent bijection between the diagrams involved in computing $(x, yy')_q$ and $(\Delta(x), y \otimes y')_q$ (using (2.1.3)). To illustrate this, consider the diagram



. (2.1.10)

It can be thought of as either a diagram computing $(x, yy')_q$ in the usual way or as a diagram $(\Delta(x), y \otimes y')_q$ as follows: there are two separated parts on bottom representing y and y'. Crossings among y-strands and among y'-strands represent crossings in diagrams used to compute $(\Delta(x)_{(1)}, y)_q$ and $(\Delta(x)_{(2)}, y')_q$. Crossings between these two groupings give the signs occurring in the formula for $\Delta(x)$ (q-algebra homomorphism property). Thus there is the claimed bijection and that bijection is compatible with signs.

For example, if x has a factor of h_2, then $\Delta(x)$ has a factor of $h_2 \otimes 1 + h_1 \otimes h_1 + 1 \otimes h_2$. The three terms of $\Delta(h_2)$ represent the cases in which both strands from the h_2 in x go to y, one strand goes to each of y and y', and both strands go to y'. $\square$

Notation: Suppose A is a $\Bbbk$-algebra and $\langle \cdot, \cdot \rangle$ is a nondegenerate bilinear form on A. Then for any element $f \in A$, the $\Bbbk$-linear map $f^\perp : A \to A$ is defined by $\langle f^\perp(g), h\rangle = \langle g, fh\rangle$ for all $g, h \in A$.

Proposition 2.1.3. 1. The coproduct of an elementary function is given by

$$\Delta(e_k) = \sum_{i=0}^{k} e_i \otimes e_k. \tag{2.1.11}$$

2. If $\alpha \vDash k$, then

$$(h_\alpha, e_k)_q = \begin{cases} 1 & \text{if } \alpha = (1, \dots, 1) \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.12}$$

3. $(e_k, e_k)_q = q^{-\binom{k}{2}}$.

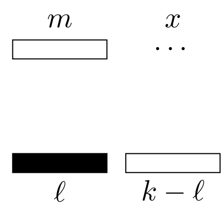
Proof. We will prove (1) and (2) simultaneously by induction on the degree k (the cases $k = 0, 1$ are clear); from these, (3) will follow. To prove the second statement, it suffices to prove that for $m > 0$,

$$h_m^\perp(e_k) = \begin{cases} e_{k-1} & \text{if } m = 1, \\ 0 & \text{otherwise} \end{cases}$$

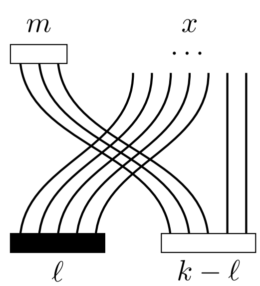
for any $x \in N\Lambda^q$. First, a calculation: for $\ell < k$, the inductive hypothesis implies

$$(h_m x, e_\ell h_{k-\ell})_q = q^{\ell m}(x, e_\ell h_{k-\ell-m})_q + q^{(\ell-1)(m-1)}(x, e_{\ell-1} h_{k-\ell-m+1})_q. \tag{2.1.13}$$

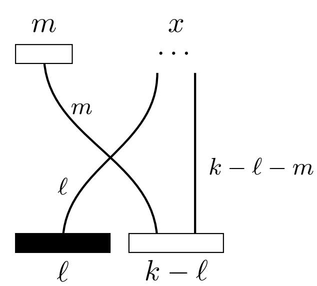
To derive this equation, we first have to enrich the bilinear form diagrammatics: we represent h_k's by white platforms of size k and e_k's by black platforms of size k. Now we start proving (2.1.13) by drawing $e_\ell h_{k-\ell}$ below $h_m x$.



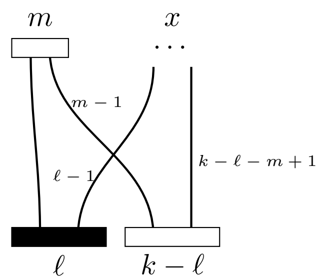
By the inductive hypothesis applied to $\ell < k$, at most one line can connect the bottom left black platform of width ℓ (representing e_ℓ) with the top left white platform of width m (representing h_m). If no lines connect these two platforms, all lines from h_m will be connected to $h_{k-\ell}$ (necessarily requiring $\ell - k \geq m$), while all lines from e_ℓ will go into x, creating ℓm intersection points that contribute $q^{\ell m}$; see below. The contribution from these diagrams will total $q^{\ell m}(x, e_\ell h_{k-\ell-m})_q$. The region immedaitely around the ellipsis representing x produce the factor $(x, e_\ell h_{k-\ell-m})_q$.



, or simply



If one line connects the e_ℓ with the h_m platform, the remaining $\ell - 1$ lines from the black platform go into x, while $m - 1$ lines from h_m enter $h_{k-\ell}$. These two types of lines intersect and contribute $q^{(\ell-1)(m-1)}$ to the sum. In the diagram below we denote each of these bunches of "parallel" lines by a single line labelled $\ell - 1$, respectively $m - 1$. The dotted curve below encloses the area contributing the factor $(x, e_{\ell-1} h_{k-m-\ell+1})_q$.



This computation proves (2.1.13). Therefore

$$\begin{aligned}
-(-1)^k q^{-\binom{k}{2}}(h_m x, e_k)_q &\overset{(2.1.7)}{=} \sum_{k=0}^{k-1}(-1)^\ell q^{\binom{\ell}{2}}(h_m x, e_\ell h_{k-\ell})_q \\
&\overset{(2.1.13)}{=} \sum_{\ell=0}^{k-1}(-1)^\ell q^{\binom{\ell}{2}}\left[q^{\ell m}(x, e_\ell h_{k-\ell-m})_q + q^{(\ell-1)(m-1)}(x, e_{\ell-1}h_{k-\ell-m+1})_q\right] \\
&= (-1)^{k-1}q^{\binom{k-1}{2}+(k-1)m}(x, e_{k-1}h_{1-m})_q.
\end{aligned}$$

The third equality follows because all terms but one cancel in pairs. Since $h_i = 0$ for $i < 0$, the second statement follows. The first statement follows from the second, since adjointness of multiplication and comultiplication implies (recall that the bilinear form is symmetric)

$$(\Delta(e_k), h_\lambda \otimes h_\mu)_q = (e_k, h_\lambda h_\mu)_q = \begin{cases} 1 & \lambda = (1^\ell), \mu = (1^p), \ell + p = k, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the third statement follows from sovling the recursion relation which results from the calculation

$$\begin{aligned}
-(-1)^k q^{\binom{k}{2}}(e_k, e_k)_q &\overset{(2.1.7)}{=} \sum_{i=0}^{k-1}(-1)^i q^{\binom{i}{2}}(e_k, e_i h_{k-i})_q \\
&\overset{(2.1.12)}{=} (-1)^{k-1}q^{\binom{k-1}{2}}(e_k, e_{k-1}h_1)_q \\
&= (-1)^{k-1}q^{\binom{k-1}{2}}(\Delta(e_k), e_{k-1}\otimes h_1)_q \\
&\overset{(2.1.11),(2.1.12)}{=} (-1)^{k-1}q^{\binom{k-1}{2}}(e_{k-1}, e_{k-1})_q.
\end{aligned}$$

$\square$

It is immediate from the discussion of the bilinear form above that in both $N\Lambda^q$ and Λ^q,

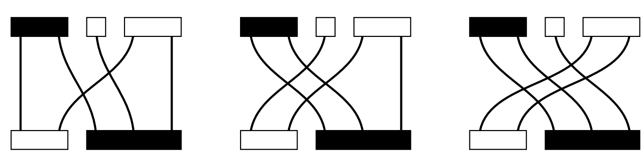
$$h_k^\perp(h_\ell) = h_{\ell-k}, \qquad h_k^\perp(e_\ell) = \begin{cases} e_{\ell-k} & k = 0,1 \\ 0 & \text{otherwise,} \end{cases}$$

$$e_k^\perp(e_\ell) = q^{\binom{k}{2}} e_{\ell-k}, \qquad e_k^\perp(h_\ell) = \begin{cases} h_{\ell-k} & k = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

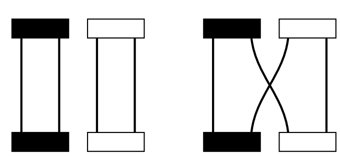
Remark 2.1.4. Proposition 2.1.3 can be used to extend our diagrammatics for $(\cdot,\cdot)_q$ so as to handle e_k's as well. Let $\alpha = (\alpha_1, \ldots, \alpha_r)$, $\beta = (\beta_1, \ldots, \beta_s)$ be compositions of k. For each $1 \leq i \leq r$ and each $1 \leq j \leq s$ choose "h" or "e". Take the minimal double coset representatives for $S_\alpha \backslash S_k / S_\beta$ as before, but now draw h_k's as white platforms (as above) and e_k's as black platforms. The sign of a diagram is computed as follows:

- for each crossing, accrue a factor of q
- if two black platforms are connected by ℓ strands, accrue a factor of $q^{-\binom{\ell}{2}}$
- if a white and a black platform are connected by more than one strand, accrue a factor of 0 (the diagram does not count)

Example 2.1.5. $(e_2h_1h_2, h_2e_3)_q = q^2 + q^3 + q^5$.



Example 2.1.6. $(e_2h_2, e_2h_2)_q = q^{-1} + q$.



It follows from equation (2.1.7) that each e_k can be written as a word in the h_i's, so every element of $N\Lambda_q$ is a word in the e_k's as well. It follows from the results of [GKL+95] that these elementary functions form a basis as well: $\{e_\alpha\}_{\alpha \vDash k}$ is a basis for $N\Lambda_q$ in degree k.

We introduce three symmetries of $N\Lambda^q$.

$$\begin{aligned}
&\psi_1(h_n) = e_n \\
&\qquad q\text{-bialgebra automorphism,} \\
&\psi_2(h_n) = (-1)^n q^{\binom{n}{2}} h_n \\
&\qquad \text{algebra automorphism (not a coalgebra homomorphism),} \\
&\psi_3(h_n) = h_n \\
&\qquad \text{algebra anti-involution (not a coalgebra homomorphism).}
\end{aligned} \tag{2.1.14}$$

Proposition 2.1.7. Let $S = \psi_1\psi_2\psi_3$. Then with the (co)multiplication and (co)unit already defined and with S as antipode, $N\Lambda^q$ has the structure of an involutory $\mathbb{Z}$-graded q-Hopf algebra. We have

$$\begin{aligned}
\psi_1\psi_2(h_\alpha) = (-1)^{|\alpha|} q^{\binom{\alpha}{2}} e_\alpha, \qquad \psi_1\psi_2(e_\alpha) = (-1)^{|\alpha|} q^{-\binom{\alpha}{2}} h_\alpha, \\
S(h_\alpha) = (-1)^{|\alpha|} q^{\binom{\alpha}{2}} e_{\alpha^{\mathrm{rev}}}, \qquad S(e_\alpha) = (-1)^{|\alpha|} q^{-\binom{\alpha}{2}} h_{\alpha^{\mathrm{rev}}}.
\end{aligned} \tag{2.1.15}$$

In the setting of ordinary vector spaces, if a bialgebra admits a Hopf antipode then this antipode is unique. The same is true in more general settings including ours; see [Maj91].

Proof. Letting η be the unit and ϵ the counit, $(\eta \circ \epsilon)(h_\lambda) = \delta_{\lambda,(0)}$. And

$$(m \circ (S \otimes 1) \circ \Delta)(h_\alpha) = \prod_{i=1}^{\ell(\alpha)} \left(\sum_{k=0}^{\alpha_i} (-1)^k q^{\binom{k}{2}} e_k h_{\alpha_i - k} \right) = \prod_{i=1}^{\ell(\alpha)} \delta_{\alpha_i,0} = \delta_{\alpha,(0)}.$$

as well. The same is true of $m \circ (1 \otimes S) \circ \Delta$, since $\Delta(h_n)$ is invariant under the map which swaps its tensor factors (without factors of q). So S is the Hopf antipode.

The expressions for $\psi_1\psi_2(h_\alpha)$ and $S(h_\alpha)$ in equation (2.1.15) are immediate from the definitions of ψ_1, ψ_2, ψ_3 and the above calculation. In order to compute $\psi_1\psi_2(e_n)$ and $S(e_n)$, we proceed by induction (the $n = 1$ case is clear). Applying ψ_3 to equation (2.1.7), we have

$$\sum_{i=0}^{k} (-1)^i q^{\binom{i}{2}} h_{k-i} e_i = 0. \tag{2.1.16}$$

Hence

$$(-1)^k q^{\binom{k}{2}} \psi_1\psi_2(e_k) \overset{(2.1.7)}{=} -\sum_{i=0}^{k-1} (-1)^i q^{\binom{i}{2}} \psi_1\psi_2(e_i h_{k-i}) \quad (2.1.17)$$

$$= -\sum_{i=0}^{k-1} (-1)^{k-i} q^{\binom{k-i}{2}} h_i e_{k-i} \quad (2.1.18)$$

$$\overset{(2.1.16)}{=} h_k. \quad (2.1.19)$$

The second equality is by the inductive hypothesis. Inserting a ψ_3 inside the $\psi_1\psi_2$ changes nothing in this calculation. Since $\psi_1\psi_2$ and S are (anti-)homomorphisms, this immediately generalizes to prove the expressions for $\psi_1\psi_2(e_\alpha)$ and $S(e_\alpha)$ in equation (2.1.15). □

Corollary 2.1.8. We have $\psi_2\psi_1\psi_2 = \psi_1^{-1}$ and $\psi_1\psi_2\psi_1 = \psi_2^{-1}$.

Proof. Immediate from the preceding proposition. □

2.1.3 q-symmetric functions

Proposition 2.1.9. The radical R^q of $(\cdot,\cdot)_q$ is a graded q-Hopf ideal.

Proof. This is apparent from the definition of $(\cdot,\cdot)_q$, the q-Hopf structre on $N\Lambda^q$, and some meditation upon the diagram in (2.1.10). □

Definition 2.1.10. The *q-symmetric functions* are defined to be the q-Hopf algebra

$$\Lambda^q = N\Lambda^q / R^q. \quad (2.1.20)$$

Regarding the size of Λ^q:

- For generic q (neither a root of unity nor certain sporadic algebraic integers), $R^q = 0$ and $\Lambda^q \cong N\Lambda^q$.

- For $q = 1$, the radical R^1 is generated as an algebra ideal by all commutators $h_a h_b - h_b h_a$. Hence $\Lambda^1 \cong \Lambda$, the usual symmetric functions (commutative, cocommutative). This is Proposition 2.1.1.

- For $q = -1$, the Hopf superalgebra Λ^{-1} has the same graded rank of Λ but is neither supercommutative nor supercocommutative. See Propsition 2.2.2 in the following section for the explicit relations.

- For $q \neq \pm 1$ a root of unity, R^q is nonzero but of lesser graded rank than $R^{\pm 1}$ [Zag92]. Very few explicit relations are known in this case.

The q-Hopf algebra $Q\Lambda_q$ of *quantum quasisymmetric functions*, first defined in [TU96], is graded dual to $N\Lambda_q$. While we will make no direct use of $Q\Lambda_q$, the fact that $Q\Lambda_q$ is a subalgebra of Pol^q (q-commuting power series of bounded degree) means that we can study Λ^q from the perspective of q-polynomials. The usual pairing between $N\Lambda^q$ and $Q\Lambda^q$ is the one making the complete basis of $N\Lambda^q$ dual to the monomial basis of $Q\Lambda^q$; or, equivalently, the ribbon basis of $N\Lambda^q$ dual to the fundamental basis of $Q\Lambda^q$ (see [Thi01] for definitions of these bases). Under the duality φ induced by this pairing, h_k and e_k are sent to the familiar expressions

$$\varphi(h_k) = \sum_{i_1 \leq \ldots \leq i_k} x_{i_1} \cdots x_{i_k}, \qquad \varphi(e_k) = \sum_{i_1 < \ldots < i_k} x_{i_1} \cdots x_{i_k}. \tag{2.1.21}$$

We have a diagram

$$\begin{tikzcd} N\Lambda^q \arrow[dr, two heads] & & Q\Lambda^q \\ & \Lambda^q \arrow[ur, hook] & \end{tikzcd}$$

in which reflection about the central vertical axis is graded Hopf duality.

2.2 Odd symmetric functions

2.2.1 Relations and basic properties

From now on, $q = -1$.

We refer to this as the "odd setting" and the $q = 1$ case as the "even setting." We will also drop the subscript from the bilinear form.

Definition 2.2.1. Let $O\Lambda = \Lambda^{-1}$, the Hopf superalgebra of *odd symmetric functions.*

Proposition 2.2.2. The radical R^{-1} is generated, as an algebra ideal, by the following relations:

$$h_a h_b = h_b h_a \text{ if } a+b \text{ is even,} \tag{2.2.1}$$

$$h_a h_b + (-1)^a h_b h_a = (-1)^a h_{a+1} h_{b-1} + h_{b-1} h_{a+1} \text{ if } a+b \text{ is odd.} \tag{2.2.2}$$

A basis for $O\Lambda$ in degree k is given by $\{h_\lambda\}_{\lambda \vdash k}$.

When $a = 2k$ is even and $b = 1$, the relation takes the form

$$h_{2k} h_1 + h_1 h_{2k} = 2h_{2k+1}.$$

Proof. We prove both relations simultaneously by induction on the total degree $a+b$. First suppose $a+b$ is even. It suffices to prove $(h_a h_b - h_b h_a, e_k x) = 0$ for $k = 1, 2$ since the products $(h_a h_b, e_k x)$ and $(h_b h_a, e_k x)$ equal zero for any $k > 2$ by Proposition 2.1.3. Computing graphically,

$$\begin{aligned} (h_a h_b, e_1 x) &= (h_{a-1} h_b, x) + (-1)^a (h_a h_{b-1}, x), \\ (h_a h_b, e_2 x) &= (-1)^{a-1} (h_{a-1} h_{b-1}, x), \end{aligned} \tag{2.2.3}$$

and likewise for $(h_b h_a, e_k x)$. When $k = 1$ the difference $(h_a h_b - h_b h_a, e_1 x)$ vanishes by the odd degree relation in degree $a+b-1$, and when $k = 2$ the difference vanishes by the even degree relation in degree $a+b-2$.

For $a+b$ odd, put together terms for $h_a h_b$, $(-1)^a h_b h_a$, $(-1)^a h_{a+1} h_{b-1}$, and $h_{b-1} h_{a+1}$ as in equation (2.2.3). The result vanishes with $k = 1$ by some cancellation and the even degree relation in degree $a+b-1$, and with $k = 2$ by the odd degree relation in degree $a+b-2$. $\square$

Proposition 2.2.3. The following relations hold in $O\Lambda$:

$$e_a e_b = e_b e_a \text{ if } a+b \text{ is even,} \tag{2.2.4}$$

$$e_a e_b + (-1)^a e_b e_a = (-1)^a e_{a+1} e_{b-1} + e_{b-1} e_{a+1} \text{ if } a+b \text{ is odd.} \tag{2.2.5}$$

$$h_a e_b = e_b h_a \text{ if } a+b \text{ is even,} \tag{2.2.6}$$

$$h_a e_b + (-1)^a e_b h_a = (-1)^a h_{a+1} e_{b-1} + e_{b-1} h_{a+1} \text{ if } a+b \text{ is odd.} \tag{2.2.7}$$

Proof. The first two can be proved by the same argument as for Proposition 2.2.2. The proof of the second two is along similar lines, but with the slight complication that terms with $k > 2$ do contribute. As in that proof, we prove both relations simultaneously by induction on the total degree $a + b$.

For $a + b$ even, we compute

$$\begin{aligned}(h_a e_b, e_k x) &= (h_{a-k} e_b + (-1)^{b-k+1} h_{a-k+1} e_{b-1}, x),\\ (e_b h_a, e_k x) &= ((-1)^{kb} e_b h_{a-k} + (-1)^{(k-1)(b-1)} e_{b-1} h_{a-k+1}, x).\end{aligned}$$

We want to show that the difference of the two left arguments on the right-hand side is zero. For k even, this difference vanishes by applying (2.2.6) twice in degree $a + b - k$. For k odd, it vanishes by applying (2.2.7).

For $a + b$ odd, we compute

$$\begin{aligned}(h_a e_b, h_k x) &= (h_{a-k} e_b + (-1)^{a-k+1} h_{a-k+1} e_{b-1}, x),\\ (e_b h_a, h_k x) &= ((-1)^{kb} e_b h_{a-k} + (-1)^{(k-1)(b-1)} e_{b-1} h_{a-k+1}, x),\\ (h_{a+1} e_{b-1}, h_k x) &= (h_{a-k+1} e_{b-1} + (-1)^{a-k} h_{a-k+2} e_{b-2}, x),\\ (e_{b-1} h_{a+1}, h_k x) &= ((-1)^{k(b+1)} e_{b-1} h_{a-k+1} + (-1)^{(k-1)b} e_{b-2} h_{a-k+2}, x).\end{aligned}$$

Considering the linear combination $h_a e_b + (-1)^a e_b h_a$, the argument which is paired with x is

$$h_{a-k} e_b - (-1)^{(k+1)b} e_b h_{a-k} + (-1)^{b-k} h_{a-k+1} e_{b-1} + (-1)^{k(b-1)} e_{b-1} h_{a-k+1}. \tag{2.2.8}$$

For the linear combination $h_{a+1} e_{b-1} + (-1)^a e_{b-1} h_{a+1}$, we get

$$(-1)^{k(b+1)} e_{b-1} h_{a-k+1} + (-1)^{b+1} h_{a-k+1} e_{b-1} + (-1)^{(k-1)b} e_{b-2} h_{a-k+2} + (-1)^k h_{a-k+2} e_{b-2}. \tag{2.2.9}$$

For k even, the difference of expressions (2.2.8) and (2.2.9) vanishes by applying (2.2.7) twice in degree $a + b - k$. For k odd, the difference vanishes by applying (2.2.6) twice. $\square$

Note that $O\Lambda$ is neither commutative, supercommutative, nor supercocommutative. It is straightforward to check that the symmetries ψ_1, ψ_2, ψ_3 all descend to $O\Lambda$ and that $\psi_2 = \psi_2^{-1}$ at $q = -1$.

Corollary 2.2.4. The degree n part of $O\Lambda$ is a free $\Bbbk$-module, of which the families $\{h_\lambda\}_{\lambda\vdash n}$ and $\{e_\lambda\}_{\lambda\vdash n}$ are both bases. Hence $O\Lambda_n$ has the same graded rank as in the even ($q = 1$) case, namely the number of partitions of n.

Proof. Any element of $O\Lambda$ is a linear combination of words in the h_k's. By the relations (2.2.4), (2.2.5), only words whose subscripts are in non-increasing order are needed; that is, $\{h_\lambda\}_{\lambda\vdash n}$ is a spanning set. Now let $O\Lambda_{\mathbb{Z}}$ be $O\Lambda$ considered over $\Bbbk = \mathbb{Z}$. Since it is expressed as the quotient of a free $\mathbb{Z}$-module by the radical of a bilinear form, $O\Lambda_{\mathbb{Z}}$ is itself a free $\mathbb{Z}$-module. The mod 2 reduction of h_λ coincides with the mod 2 reduction of the even ($q = 1$) complete symmetric function h_λ^{even}, so the spanning set $\{h_\lambda\}_{\lambda\vdash n}$ is linearly independent in $O\Lambda_{\mathbb{Z}/2}$, hence in $O\Lambda_{\mathbb{Z}}$. The same argument works for the family $\{e_\lambda\}_{\lambda\vdash n}$. Now $O\Lambda_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module with the required bases, so $O\Lambda = O\Lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$ is a free $\Bbbk$-module with the required bases. $\square$

An alternate argument deduces that $\{e_\lambda\}_{\lambda\vdash n}$ is a basis of $(O\Lambda_{\mathbb{Z}})$ in degree n from the fact that $\{h_\lambda\}_{\vdash n}$ is and the following "semi-orthogonality" property. We call a minimal double coset representative *lite* if any two platforms in its diagram are connected by at most one strand.

Proposition 2.2.5. 1. For all partitions λ,

$$(h_\lambda, e_{\lambda^T}) = (-1)^{\ell(w_\lambda)} = (-1)^{NE(\lambda)}, \tag{2.2.10}$$

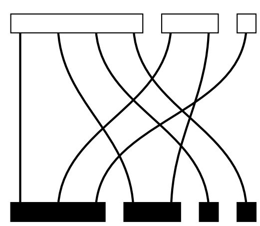
where $\ell(w_\lambda)$ is the Coxeter length of the unique lite minimal double coset representative w_λ in $S_\lambda\backslash S_n/S_{\lambda^T}$.

2. If λ is a partition and α is a composition with $\alpha > \lambda^T$ in the lexicographic order, then

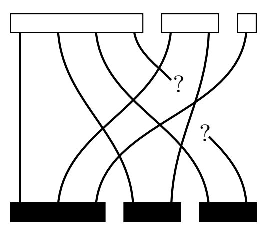
$$(h_\lambda, e_\alpha) = (e_\lambda, h_\alpha) = 0. \tag{2.2.11}$$

Remark 2.2.6. If all partitions of n are listed lexicographically, it is not true that reversing the order swaps partitions whose corresponding Young diagrams are transposes of each other (this first occurs at $n = 6$). One can, however, refine the dominance partial order in such a way that this property holds.

Proof of Proposition 2.2.5. The proof, except for the determination of the sign in equation (2.2.10), is exactly as in the case of classical symmetric functions. To compute an inner product (h_λ, e_α), we must sign-count minimal double coset representative diagrams connecting a λ-arrangement of white platforms and a α-arrangement of black platforms such that no pair of a black and a white diagram are connected by more than one strand. For $\alpha = \lambda^T$, α_i equals the number of rows of λ of size at least i; diagrammatically, the i-th black platform has exactly one strand going to each white platform of size at least i. Hence there is a unique lite diagram connecting these two platform arrangements and it counts as $(-1)^{\ell(w)}$: in this diagram, the strands of the i-th black platform go to the first α_i white platforms, and these white platforms are precisely those white platforms which accept at least i strands; and vice versa, switching black and white and switching λ and α. For example, the unique lite diagram in computing (h_{421}, e_{3211}) is:



For $\alpha > \lambda^T$, let i be minimal such that $\alpha_i > \lambda_i^T$. Then filling a potential diagram as above, at the stage of connecting the i-th black platform, there are fewer than α_i white platforms which can still accept a new strand, so we are forced to send two strands from this black platform to the same white platform (pigeonhole principle). So the diagram is zero. For example, consider the next step in filling the unfinished diagram below for $\lambda = (4, 2, 1)$, $\alpha = (3, 2, 2)$:



Connecting the strands marked "?" would result in a non-lite diagram. Finally, showing that $\ell(w_\lambda) = NE(\lambda)$ is left to the reader. $\square$

Introduce the notation

$$H_{\geq\lambda} = \mathrm{span}_{\mathbb{Z}}\{h_\mu : \mu \geq \lambda\},$$
$$E_{\geq\lambda} = \mathrm{span}_{\mathbb{Z}}\{e_\mu : \mu \geq \lambda\}$$

and likewise with $\geq$ replaced by one of $\{\leq, >, <\}$ (lexicographic order), as subspaces of the degree n part of $O\Lambda$ for $n = |\lambda|$.

The following lemma will be useful in studying odd Schur functions.

Lemma 2.2.7. For any partition $\lambda \vdash n$, the bilinear form is nondegenerate when restricted to the subspaces $H_{\geq\lambda}$ and $E_{>\lambda^T}$ of $(O\Lambda)_n$, the degree n part of $O\Lambda$.

Proof. By equation (2.2.11) and nondegeneracy of the bilinear form, $(H_{\geq\lambda})^\perp = E_{>\lambda^T}$. If $H_{\geq\lambda} \cap E_{>\lambda^T} = \{0\}$, then it follows that

$$(O\Lambda)_n = H_{\geq\lambda} \oplus E_{>\lambda^T}$$

is an orthogonal decomposition, so that

$$\det(\cdot,\cdot)|_{H_{\geq\lambda}} \det(\cdot,\cdot)|_{E_{>\lambda^T}} = \det(\cdot,\cdot) = \pm 1,$$

which implies that both factors on the left hand side are ± 1. And since $H_{\geq\lambda} \cap E_{>\lambda^T} = \{0\}$ after reducing mod 2, the intersection must have been zero over $\mathbb{Z}$: Any nonzero element of $H_{\geq\lambda} \cap E_{>\lambda^T}$ which is zero mod 2 must be divisible by 2. But then the result of dividing this element by 2 would also be in $H_{\geq\lambda} \cap E_{>\lambda^T}$. $\square$

The previous lemma does not hold with $\geq$ replaced by $\leq$. For instance, $(h_{11}, h_{11}) = 0$.

By definition, $\psi_3(h_n) = h_n$. The characterization of e_n in equation (2.1.12) and the fact that ψ_3 is norm preserving imply $\psi_3(e_n) = e_n$. Of course, this does not extend to other partitions λ. For instance,

$$\psi_3(h_2h_1) = h_1h_2 = 2h_3 - h_2h_1.$$

But ψ_3 does preserve e_λ, h_λ up to lexicographically higher order terms.

Lemma 2.2.8. $\psi_3(h_\lambda)$ is in $H_{\geq\lambda}$, and the coefficient of h_λ in $\psi_3(h_\lambda)$ (when expanding in the complete basis) is computed as follows: Write the row lengths of λ in reverse order.

In permuting these to get λ again, accrue a -1 each time an odd number on the left is transposed with an even number on the right. Furthermore, the same holds with h, H replaced by e, E and "complete" replaced by "elementary" (no transpositions of diagrams are necessary).

Proof. Consider all compositions of a fixed degree to be ordered lexicographically. By induction, then, it suffices to show that whenever $a < b$, $h_a h_b$ is in $H_{\geq(b,a)}$. If $a + b$ is even, then $h_a h_b = h_b h_a$ and we are done. If $a + b$ is odd, apply the odd degree h-relation:

$$h_a h_b = (-1)^a h_b h_a + h_{b+1} h_{a-1} - (-1)^a h_{a-1} h_{b+1}.$$

The first and second terms on the right-hand side are now lexicographically greater than $h_a h_b$ and in non-increasing order, so it remains to express $h_{a-1} h_{b+1}$ as a linear combination of terms lexicographically higher. To do so, apply the odd degree h-relation to $h_{a-1} h_{b+1}$, and then to the last term in that, and so forth until the left factor's subscript reaches one. At this point apply $h_1 h_{a+b-1} = 2h_{a+b} - h_{a+b-1} h_1$, and we are done. Going through the algorithm just described, the sign of the h_λ term in $\psi_3(h_\lambda)$ is clearly as in the statement of lemma. $\square$

2.2.2 Dual bases

In the even ($q = 1$) case, the dual bases to the elementary and complete symmetric functions are the forgotten and the monomial symmetric functions, respectively. The monomial functions $\{m_\lambda\}$ get their name from the fact that when Λ is viewed in terms of power series, they are sums of monomials of the same shape,

$$m_\lambda = \sum_\alpha x^\alpha. \tag{2.2.12}$$

Here, $\lambda = (\lambda_1, \ldots, \lambda_r, 0, \ldots)$ is a partition padded with infinitely many zeroes at the end, the sum ranges over all *distinct* permutations α of λ, and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. In terms of power series, no particularly nice description of the forgotten symmetric functions is known. As a result they are often omitted from the discussion; whence their name. From the point of view of self-adjoint Hopf algebras with a bilinear form, however, they are just as natural a consideration as the monomial functions; see [Mac95, I.2].

We now return to the odd ($q = -1$) case. For $n \geq 0$, define the *odd monomial symmetric functions* $\{m_\lambda\}_{\lambda \vdash n}$ to be the dual basis to $\{h_\lambda\}_{\lambda \vdash n}$ and define the *odd forgotten symmetric functions* $\{f_\lambda\}_{\lambda \vdash n}$ to be the dual basis to $\{e_\lambda\}_{\lambda \vdash n}$. In other words, we define m_λ and f_λ by the conditions

$$(h_\lambda, m_\mu) = \delta_{\lambda\mu}, \qquad (e_\lambda, f_\mu) = \delta_{\lambda\mu}.$$

Define the coefficients $M_{\lambda\mu}, M'_{\lambda\mu}, M''_{\lambda\mu}$ (indexed over ordered pairs of partitions λ, μ of some n) by

$$M_{\lambda\mu} = (e_\lambda, h_\mu), \qquad M'_{\lambda\mu} = (h_\lambda, h_\mu), \qquad M''_{\lambda\mu} = (e_\lambda, e_\mu). \tag{2.2.13}$$

The following change of basis relations are immediate consequences of (2.2.13):

$$\begin{aligned} h_\lambda &= \sum_{\mu \vdash n} M_{\lambda\mu} f_\mu, & h_\lambda &= \sum_{\mu \vdash n} M'_{\lambda\mu} m_\mu, \\ e_\lambda &= \sum_{\mu \vdash n} M_{\lambda\mu} m_\mu, & e_\lambda &= \sum_{\mu \vdash n} M''_{\lambda\mu} f_\mu. \end{aligned} \tag{2.2.14}$$

Along with the results of Subsection 2.2.1, we see that these change of basis matrices have the properties:

- $M_{\lambda\mu}$ is equal to 0 when $\mu > \lambda^T$ in the lexicographic order and equal to ± 1 when $\mu = \lambda^T$. So the change of basis matrix is upper-left-triangular with ± 1's on the diagonal.

- The matrix for $M'_{\lambda\mu}$ is symmetric and has determinant equal to ± 1.

- The matrix for $M''_{\lambda\mu}$ is symmetric and has determinant equal to ± 1.

Since $(e_\lambda, e_\mu) = (h_\lambda, h_\mu)$ when $q = 1$, the $q = 1$ analogues of $M'_{\lambda\mu}$ and $M''_{\lambda\mu}$ are equal. Their combinatorial interpretations in Proposition 2.2.9 below are the same when the signs are omitted. In the odd ($q = -1$) case, they differ because $(e_k, e_k) = (-1)^{\binom{k}{2}}$, as this sign comes up whenever k strands connect the same two black platforms.

The determinant of the matrix M is not hard to compute. M is upper-left-triangular by Proposition 2.2.5, and the anti-diagonal entry $(h_\lambda, e_{\lambda^T})$ equals $(-1)^{NE(\lambda)}$. But this entry and the anti-diagonal entry $(h_{\lambda^T}, e_\lambda)$ are equal, so the determinant of M in degree n is a sign computed only from self-transpose diagrams:

$$\det(M_n) = \prod_{\lambda = \lambda^T} (-1)^{NE(\lambda)}. \tag{2.2.15}$$

The determinants $\det(M'_n)$ and $\det(M''_n)$ both equal $\det(M_n)$ times the determinant of the change of basis between the e- and h-bases. Note that self-transpose Young diagrams with n boxes are in a natural bijection with partitions of n into distinct odd positive integers. Under this bijection, the sign $(-1)^{NE(\lambda)}$ has a factor of -1 for each summand which is congruent to 3 modulo 4.

The proof of the following proposition, which we omit, is essentially the same as in the even ($q = 1$) case, but with the extra bookkeeping of signs. For the even case, see Proposition 37.5 of [Bum04]. For a matrix A, define the composition $\mathrm{row}(A)$ (respectively $\mathrm{col}(A)$) to consist of the row (respectively column) sums of A. If A is an $\mathbb{Z}_{\geq 0}$-matrix (that is, its entries are all natural numbers), there are two sorts of signs we can attach to A when counting matrices. Our natural numbers include zero: $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$.

- To count *SW-NE pairs* means to accrue a sign of $(-1)^{ab}$ for every pair of entries in which an a is strictly below and strictly to the left of a b.
- To count *cables* means to accrue a sign of $(-1)^{\binom{a}{2}}$ for every entry a. Since 0- and 1-cables accrue 1's, this is not interesting for $\{0, 1\}$-matrices.

Proposition 2.2.9. The numbers defined in equation (2.2.13) have the following combinatorial interpretations:

1. $M_{\lambda\mu}$ equals the signed count of $\{0, 1\}$-matrices A with $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$. The sign counts SW-NE pairs.
2. $M'_{\lambda\mu}$ equals the signed count of $\mathbb{Z}_{\geq 0}$-matrices A with $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$. The sign counts SW-NE pairs.
3. $M''_{\lambda\mu}$ equals the signed count of $\mathbb{Z}_{\geq 0}$-matrices A with $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$. The sign counts SW-NE pairs and cables.

Example 2.2.10. We compute the $(3, 2), (2, 2, 1)$ entry of the matrices M, M', M''. Below are the five $\mathbb{Z}_{\geq 0}$-matrices with row sum $(3, 2)$ and column sum $(2, 2, 1)$, and their contributions to M, M', M''.

matrix	contribution to M	contribution to M'	contribution to M''
$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	0	$(-1)^0$	$(-1)^0(-1)^{\binom{3}{2}}$
$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$	0	$(-1)^2$	$(-1)^2(-1)^{\binom{3}{2}+\binom{3}{2}}$
$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	0	$(-1)^2$	$(-1)^2(-1)^{\binom{3}{2}}$
$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	$(-1)^3$	$(-1)^3$	$(-1)^3(-1)^0$
$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix}$	0	$(-1)^6$	$(-1)^6(-1)^{\binom{3}{2}+\binom{3}{2}}$

Therefore

$$M_{(3,2),(2,2,1)} = -1, \qquad M'_{(3,2),(2,2,1)} = 3, \qquad M''_{(3,2),(2,2,1)} = -1.$$

We end this section by pointing out that the above results are enough to compute the matrix of the bilinear form in any of the bases described so far. For instance, since M' is the matrix of the bilinear form in the h-basis, M is the matrix which takes the f-basis to the h-basis, and $M = M^T$, the matrix $M^{-1}M'M^{-1}$ is the matrix of the bilinear form in the f-basis.

2.2.3 The puzzle of primitives

For this section assume $\Bbbk$ is a field of characteristic zero.

Recall that an element x of a (q-)Hopf algebra is called *primitive* if

$$\Delta(x) = 1 \otimes x + x \otimes 1.$$

In the even ($q = 1$) case, the primitive elements of Λ are spanned by the power sum functions

$$p_n = \sum_j x_j^n.$$

In the odd setting, however, there are only "half" as many.

Proposition 2.2.11. The subspace of primitive elements P in $O\Lambda$ is spanned by the elements m_1 and m_{2k} for $k \geq 1$.

Proof. Let $I \subset O\Lambda$ be the ideal generated by all elements of positive degree. By the general theory of self-adjoint connected graded Hopf algebras with a bilinear form, $P = (I^2)^\perp$ (Lemma 1.7 of [Zel81]). It is clear from the h-relations and the e-relations that in each degree n,

$$I^2 \cap (O\Lambda)_n = \begin{cases} 0 & n = 1, \\ \mathrm{span}_{\Bbbk}\{h_\lambda : \lambda \neq (n)\} & n \text{ is even}, \\ \Lambda_n & n \text{ is odd and } \geq 3. \end{cases}$$

The result follows. ☐

Note that $f_n = \pm m_n$, so the f_{2k} are primitive as well (the sign is the same as the coefficient of h_n in the expansion of e_n in the h-basis). We define, therefore, the n-th *odd power symmetric function* to be

$$p_n = m_n.$$

Proposition 2.2.12. The element p_k belongs to the center of Λ if and only if k is even.

Proof. We will show that $(p_k h_m, e_\lambda) = (h_m p_k, e_\lambda)$ for every $m \geq 0$ and every $\lambda \vdash (k+m)$, if and only if k is even. Let ℓ be the length of λ. The coproduct of e_λ is

$$\Delta(e_\lambda) = \prod_{i=1}^{\ell} \sum_{j=0}^{\lambda_i} e_j \otimes e_{\lambda_i - j} = \sum_\alpha (e_{a_1} \otimes e_{\lambda_1 - a_1}) \cdots (e_{a_\ell} \otimes e_{\lambda_\ell - a_\ell}),$$

where the last sum is over all α such that $|\alpha| = k + m$ and $0 \leq a_j \leq \lambda_j$ for each j. When paired against $p_k h_m$ or $h_m p_k$, only partitions $\lambda = (k+1, 1^{m-1})$ and $\lambda = (k, 1^m)$ yield nonzero results. It is straightforward to check that

$$(p_k h_m, e_{k+1} e_1^{m-1}) = 1, \qquad (h_m p_k, e_{k+1} e_1^{m-1}) = (-1)^{k(m-1)},$$
$$(p_k h_m, e_k e_1^m) = 1, \qquad (h_m p_k, e_k e_1^m) = (-1)^{km},$$

using the adjointness of multiplication and comultiplication. The result follows. ☐

In fact, the center is precisely the polynomial algebra generated by the p_{2k}'s. This will follow from Corollary 3.2.7 below.

Remark 2.2.13. In the next chapter, we will realize p_k as a power sum $\sum_j x_j^k$ when $k = 1$ or $k \geq 2$ is even. The corresponding power sums $\sum_j x_j^k$ for $k \geq 3$ and odd are not odd symmetric, nor is any function obtained by inserting signs into this summation.

The lack of primitives and the resulting lack of a good analogue of power sums in all degrees makes it difficult to the mimic the usual construction of vertex operators on Λ. In the even case, one uses the Jacobi-Trudi relation, which is a determinantal formula for s_λ in terms of complete or elementary functions, to show that successive applications of creation operators to the vacuum vector yields the Schur functions. The Jacobi-Trudi relation states that

$$s_\lambda^{\text{even}} = \det(h_{\lambda_i+j-1}).$$

For example, $s_{22}^{\text{even}} = h_2^2 - h_3h_1$. In the odd setting, one piece of evidence for the impossibility of any naïve analogue of this process is the fact that the corresponding odd Schur function,

$$s_{22} = h_2^2 + h_3h_1 - 2h_4,$$

is not even in the subalgebra of $O\Lambda$ generated by h_1, h_2, and h_3.

2.2.4 Odd Schur functions

In the even ($q = 1$) case, the Schur functions $\{s_\lambda\}_{\lambda\vdash n}$ form an orthonormal basis of $(\Lambda)_n$. In terms of power series, they are generating functions for semistandard Young tableaux. One definition of the Schur function s_λ is

$$s_\lambda = \sum_{T\in\mathrm{SSYT}(\lambda)} x^{\mathrm{ct}(T)}.$$

Then if one defines the *Kostka number* associated to partitions λ, μ to be

$$K_{\lambda\mu} = \#SSYT(\lambda,\mu),$$

it follows from (2.2.12) that

$$s_\lambda = \sum_{\mu\vdash n} K_{\lambda\mu} m_\mu. \tag{2.2.16}$$

Having expressed Schur functions in terms of the dual basis to the complete functions, we have a definition which we can attempt to mimic in the odd ($q = -1$) case. An essential feature in the even case is that the Schur functions $\{s_\lambda\}_{\lambda \vdash n}$ form an orthonormal basis of $(\Lambda)_n$; in the odd case, Schur functions will be orthogonal but their norms may be either 1 or -1.

In the odd case, we define the *odd Schur functions* by a change of basis relation closely related to (2.2.16),

$$h_\mu = \sum_{\lambda \vdash n} K_{\lambda\mu} s_\lambda. \tag{2.2.17}$$

To define the coefficients $K_{\lambda\mu}$, the *odd Kostka numbers*, we first define the sign associated to a Young tableau T. For a Young tableau T, let $w_r(T)$ be its *row word*, that is, the string of numbers obtained by reading the entries of T from left to right, bottom to top. Then we count T with the sign of the minimal length permutation which sorts $w_r(T)$ into non-decreasing order; this is just $(-1)^{N^<(T)}$. For a Young diagram λ, let T_λ be the unique semistandard Young tableau with shape and content both equal to λ. In other words every first-row entry of T_λ is a 1, every second-row entry is a 2, and so forth. So as to make the matrix of odd Kostka numbers always have just 1's on the diagonal, we introduce an overall factor of $(-1)^{N(\lambda)}$. We define

$$K_{\lambda\mu} = (-1)^{N(\lambda)} \sum_T (-1)^{N^<(T)}, \tag{2.2.18}$$

where the sum is over all semistandard Young tableaux T of shape λ and content μ. Note that $K_{(n)\mu} = 1$ for all $\mu \vdash n$, $K_{(1^n)\mu} = \delta_{\mu,(1^n)}$, and $K_{\lambda\lambda} = 1$ for all λ. This definition is a straightforward generalization of Stanley's notion of the sign of a standard Young tableau, which he and others have used to study the nubmers $K_{\lambda(1^n)}$ under the name *sign imbalance* [Sta02; Lam04; Lam08; Kim10]

Example 2.2.14. To compute $K_{(2,2,1),(1^5)}$:

tableau T	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}$
$N^<(T)$	8	7	7	6	5

Since $N(2,2,1) = 8$, we have $K_{(2,2,1),(1^5)} = -1$.

Example 2.2.15. To compute $K_{(3,1,1),(2,1,1,1)}$:

tableau T	$\begin{array}{\|c\|c\|c\|}\hline 1 & 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$
$N^<(T)$	7	6	5

Since $N(3,1,1) = 7$, we have $K_{(3,1,1),(2,1,1,1)} = 1$.

The following result was originally proved by Reifegerste [Rei04] and Sjöstrand [Sjo].

Theorem 2.2.16 (Odd RSK Correspondence I)**.** The RSK map (as described in Subsection 2.2.5 is a bijection

$$\mathrm{RSK} : \left\{ \begin{array}{c} \mathbb{Z}_{\geq 0}\text{-matrices } A \text{ with} \\ \mathrm{row}(A)=\mu \text{ and } \mathrm{col}(A)=\rho \end{array} \right\} \to \left\{ \begin{array}{c} \text{pairs } (P,Q) \text{ of semistandard Young tableaux of the} \\ \text{same shape, with } \mathrm{ct}(P)=\mu \text{ and } \mathrm{ct}(Q)=\rho \end{array} \right\}, \tag{2.2.19}$$

under which the sign of A as in the computation of $M'_{\mu\rho}$ equals $(-1)^{\binom{\lambda^T}{2}+N^<(P)+N^<(Q)}$, where $\lambda = \mathrm{sh}(P) = \mathrm{sh}(Q)$. In particular,

$$\begin{aligned} M'_{\mu\rho} &= \sum_{\lambda\vdash n} (-1)^{\binom{\lambda^T}{2}} K_{\lambda\mu}K_{\lambda\rho} \\ &= \sum_{\lambda\vdash n} (-1)^{\lambda_2+\lambda_4+\lambda_6+\cdots} K_{\lambda\mu}K_{\lambda\rho}. \end{aligned} \tag{2.2.20}$$

The proof is deferred to the following subsection, where we also review the definition of the RSK map.

Corollary 2.2.17. The Schur function s_λ can be expressed in terms of the monomial functions as

$$(-1)^{\binom{\lambda^T}{2}} s_\lambda = \sum_{\mu\vdash n} K_{\lambda\mu} m_\mu. \tag{2.2.21}$$

Proof. Define matrices A, B, C, all square and indexed by all partitions of n, by

$$A_{\lambda\mu} = M'_{\lambda\mu},$$

$$B_{\lambda\mu} = K_{\lambda\mu},$$

$$C_{\lambda\mu} = (-1)^{\binom{\lambda^T}{2}} K_{\lambda\mu}.$$

The ordering on the index set can be taken to be any total ordering which refines the dominance partial order. In these terms, equation (2.2.17) says that B^T takes the Schur basis to the complete basis and equation (2.2.14) says that A takes the monomial basis to the complete basis. It follows that $(B^T)^{-1}A$ takes the monomial basis to the Schur basis. Now equation (2.2.20) says that $A = B^T C$, proving the corollary. □

Corollary 2.2.18. The Schur functions are signed-orthonormal:

$$(s_\lambda, s_\mu) = (-1)^{\binom{\lambda^T}{2}} \delta_{\lambda,\mu}. \tag{2.2.22}$$

Proof. In $\Lambda \otimes \Lambda$, equations (2.2.21) and (2.2.17) imply

$$\sum_{\lambda \vdash n} (-1)^{\binom{\lambda^T}{2}} s_\lambda \otimes s_\lambda = \sum_{\lambda,\mu \vdash n} K_{\lambda\mu} m_\mu \otimes s_\lambda = \sum_{\mu \vdash n} m_\mu \otimes h_\mu.$$

Since $\{m_\lambda\}_{\lambda \vdash n}$ and $\{h_\lambda\}_{\lambda \vdash n}$ are dual bases, it follows that $\{s_\lambda\}_{\lambda \vdash n}$ and $\{(-1)^{\binom{\lambda^T}{2}} s_\lambda\}_{\lambda \vdash n}$ are dual bases. □

In order to express the Schur functions in the elementary and forgotten bases, note that the two following properties uniquely characterize the Schur functions:

1. $(s_\lambda, h_\mu) = 0$ if $\mu > \lambda$ (lexicographic order).

2. For certain integers a_μ (depending on λ),

$$s_\lambda = h_\lambda + \sum_{\mu > \lambda} a_\mu h_\mu. \tag{2.2.23}$$

That these uniquely determine the Schur functions follows from Lemma 2.2.7. The first property follows immediately from equation (2.2.21) and the second follows from equation (2.2.17). We think of these conditions as an inductive definition of s_λ, starting from $s_{(n)} = h_n$.

Proposition 2.2.19. Define the elements s'_λ of Λ inductively as follows: $s'_{(1^n)} = e_n$, and the following two properties hold:

1. $(s'_\lambda, e_\mu) = 0$ if $\mu > \lambda^T$ (lexicographic order).

2. For certain integers b_μ (depending on λ),

$$s'_\lambda = e_{\lambda^T} + \sum_{\mu > \lambda^T} b_\mu e_\mu.$$

Then $s'_\lambda = (-1)^{NE(\lambda)+\binom{\lambda^T}{2}} s_\lambda$.

Proof. By Lemma 2.2.7 and the property (2) preceding the statement of the Proposition, the space $(H_{\geq\lambda} \cap E_{\geq\lambda^T}) \otimes \mathbb{Q}$ is one-dimensional and spanned by s_λ. But it is also spanned by s'_λ, by property (2) in the statement of the proposition. In order to determine the constant by which they differ, we compute

$$\begin{aligned}(s_\lambda, s'_\lambda) &= (h_\lambda, e_{\lambda^T}) + \sum_{\mu>\lambda} a_\mu (h_\mu, e_{\lambda^T}) \\ &\quad + \sum_{\mu > \lambda^T} b_\mu (h_\lambda, e_\mu) + \sum_{\substack{\rho>\lambda \\ \mu>\lambda^T}} a_\rho b_\mu (h_\rho, e_\mu) \\ &\overset{(2.2.11)}{=} (-1)^{NE(\lambda)}.\end{aligned}$$

Hence, by the signed orthonormality of Schur functions, $s'_\lambda = (-1)^{NE(\lambda)+\binom{\lambda^T}{2}} s_\lambda$. $\square$

Lemma 2.2.20. The anti-involution ψ_3 and the involution $\psi_1\psi_2$ act on Schur functions as follows:

$$\psi_3(s_\lambda) = (-1)^{N(\lambda)+\binom{\lambda^T}{2}} s_\lambda, \qquad \psi_1\psi_2(s_\lambda) = (-1)^{NE(\lambda)+|\lambda|} s_{\lambda^T}. \tag{2.2.24}$$

In particular, $S(s_\lambda) = (-1)^{N(\lambda)+\binom{\lambda^T}{2}+NE(\lambda)+|\lambda|} s_{\lambda^T}$.

Proof. Since ψ_3 is norm preserving, the expression for $\psi_3(s_\lambda)$ follows from Lemma 2.2.8 and the two properties stated before Proposition 2.2.19. To prove the expression for $\psi_1\psi_2(s_\lambda)$, we express s_λ in terms of both complete and elementary functions and then compare the results (a_μ, b_μ are integers depending on λ and on μ; their particular values are immaterial):

$$\begin{aligned}
\psi_1\psi_2(s_\lambda) &= \psi_1\psi_2\left(h_\lambda + \sum_{\mu>\lambda} a_\mu h_\mu\right) \\
&= (-1)^{\binom{\lambda}{2}+|\lambda|} e_\lambda + \sum_{\mu>\lambda} (-1)^{\binom{\mu}{2}+|\mu|} a_\mu h_\mu \\
\psi_1\psi_2(s_\lambda) &= \psi_1\psi_2\left((-1)^{NE(\lambda)+\binom{\lambda^T}{2}} e_{\lambda^T} + \sum_{\mu>\lambda^T} b_\mu e_\mu\right) \\
&= (-1)^{NE(\lambda)+|\lambda|} h_{\lambda^T} + \sum_{\mu>\lambda^T} (-1)^{\binom{\mu}{2}+|\mu|} b_\mu h_\mu.
\end{aligned}$$

Since $H_{\geq\lambda} \cap E_{\geq\lambda^T}$ is generated by s_{λ^T} as in the proof of Proposition 2.2.19, it follows that both the above expressions for $\psi_1\psi_2(s_\lambda)$ are equal to plus or minus s_{λ^T}. Considering the leading coefficient of either one, we see that the sign between $\psi_1\psi_2(s_\lambda)$ and s_{λ^T} must be $(-1)^{NE(\lambda)+|\lambda|}$. □

Corollary 2.2.21. The Schur function basis is related to the monomial and the complete bases as follows:

$$\begin{aligned}
(-1)^{\binom{\mu}{2}+|\mu|} e_\mu &= \sum_{\lambda\vdash n} (-1)^{NE(\lambda)+|\lambda|} K_{\lambda^T\mu} s_\lambda, \\
(-1)^{NE(\lambda)+\binom{\lambda^T}{2}+|\lambda|} s_\lambda &= \sum_{\mu\vdash n} (-1)^{\binom{\mu}{2}+|\mu|} K_{\lambda^T\mu} f_\mu.
\end{aligned} \tag{2.2.25}$$

Proof. Apply $\psi_1\psi_2$ to equations (2.2.17) and (2.2.21). □

Corollary 2.2.22 ("Odd RSK Correspondence II")**.** The following formula holds:

$$\begin{aligned}
(-1)^{\binom{\mu}{2}+|\mu|+\binom{\rho}{2}+|\rho|} M''_{\mu\rho} &= \sum_{\lambda\vdash n} (-1)^{\binom{\lambda^T}{2}} K_{\lambda^T\mu} K_{\lambda^T\rho} \\
&= \sum_{\lambda\vdash n} (-1)^{\lambda_2+\lambda_4+\lambda_6+\cdots} K_{\lambda^T\mu} K_{\lambda^T\rho}.
\end{aligned} \tag{2.2.26}$$

Proof. Argue as in the proof of Corollary 2.2.17. □

Why the scare quotes around the name of the corollary? Unlike the formula for $M'_{\mu\rho}$ (Odd RSK Correspondence I, Theorem 2.2.16), it does not appear that the above formula can be refined to a matching of signs between particular matrices and their RSK-corresponding pairs of semistandard Young tableaux. Such a refined correspondence is possible after permuting the matrices counted in a particular $M''_{\mu\rho}$, but we do not know of a general rule governing these permutations.

2.2.5 Signed RSK

Theorem 2.2.23 (RSK Correspondence, [Ful97])**.** The RSK map is a bijection

$$\mathrm{RSK} : \left\{ \begin{array}{c} \mathbb{Z}_{\geq 0}\text{-matrices } A \text{ with} \\ \mathrm{row}(A)=\mu \text{ and } \mathrm{col}(A)=\rho \end{array} \right\} \to \left\{ \begin{array}{c} \text{pairs } (P,Q) \text{ of semistandard Young tableaux of the} \\ \text{same shape, with } \mathrm{ct}(P)=\mu \text{ and } \mathrm{ct}(Q)=\rho \end{array} \right\}. \tag{2.2.27}$$

In particular,

$$N_{\mu\rho} = \sum_{\lambda\vdash n} K_{\lambda\mu} K_{\lambda\rho}. \tag{2.2.28}$$

We now describe the classical RSK map. Let A be an $\mathbb{Z}_{\geq 0}$-matrix such that $\mathrm{row}(A) = \mu$ and $\mathrm{col}(A) = \rho$ are partitions and note that the sum of the entries of A equals n. For the purposes of this discussion, consider an entry equal to k to be k distinct entries, each equal to 1, all in the same place. Order the entries A from left to right and top to bottom, as if reading a book. For $j = 1, \ldots, n$, let u_j be the row number and v_j the column number of the j-th entry in this ordering. Organize these coordinates into a two-line array,

$$\begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix} = \begin{pmatrix} u_1, u_2, \ldots u_n \\ v_1, v_2, \ldots, v_n \end{pmatrix}.$$

Note that $\underline{u}$ will always be non-decreasing.

To this two-line array, we associate a pair (P, Q) of tableaux in the following way. For $k = 1, \ldots, n$, let P_k be the unique tableau whose row word $w_r(P_k)$ is Knuth equivalent to $v_1 \cdots v_k$, as guaranteed by Theorem 3.7.1. This is equivalent to proceeding one box at a time in Schensted's row insertion (bumping) algorithm. So at each step, P_k is a Young tableau with k boxes whose entries are $\{v_1, \ldots, v_k\}$, and the shape of P_k is obtained from the shape of P_{k-1} by adding one box. Let Q_1 be the one-box Young tableau with entry u_1. Inductively, build Q_k from Q_{k-1} by placing a new box with entry u_k at the location of the box of P_k which was not in P_{k-1}. So at each step, P_k and Q_k have the same shape, and the entries of Q_k are $\{u_1, \ldots, u_k\}$.

The RSK map assigns the pair of final tableaux (P_n, Q_n) to the matrix A.

Example 2.2.24. We illustrate the RSK correspondence and equation (2.2.28) for $\mu = \ydiagram{1,1,1}$ and $\rho = \ydiagram{2,1}$.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow \left(\ytableaushort{112}, \ytableaushort{123}\right), \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \leftrightarrow \left(\ytableaushort{11,2}, \ytableaushort{12,3}\right), \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \leftrightarrow \left(\ytableaushort{11,2}, \ytableaushort{13,2}\right).$$

Indeed,

$$N_{\ydiagram{1,1,1},\ydiagram{2,1}} = 3, \qquad K_{\ydiagram{3},\ydiagram{1,1,1}} K_{\ydiagram{3},\ydiagram{2,1}} = 1 \cdot 1, \qquad K_{\ydiagram{2,1},\ydiagram{1,1,1}} K_{\ydiagram{2,1},\ydiagram{2,1}} = 2 \cdot 1, \qquad K_{\ydiagram{1,1,1},\ydiagram{2,1}} K_{\ydiagram{1,1,1},\ydiagram{1,1,1}} = 0 \cdot 1.$$

Example 2.2.25. We illustrate the RSK correspondence and equation (2.2.28) for $\mu = \ydiagram{2,2}$ and $\rho = \ydiagram{2,1,1}$.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \leftrightarrow \left(\ytableaushort{1123}, \ytableaushort{1122}\right), \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \leftrightarrow \left(\ytableaushort{113,2}, \ytableaushort{112,2}\right),$$
$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \leftrightarrow \left(\ytableaushort{112,3}, \ytableaushort{112,2}\right), \quad \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix} \leftrightarrow \left(\ytableaushort{11,23}, \ytableaushort{11,22}\right).$$

Indeed,

$$N_{\ydiagram{2,2},\ydiagram{2,1,1}} = 4, \qquad K_{\ydiagram{4},\ydiagram{2,2}} K_{\ydiagram{4},\ydiagram{2,1,1}} = 1 \cdot 1, \qquad K_{\ydiagram{3,1},\ydiagram{2,2}} K_{\ydiagram{3,1},\ydiagram{2,1,1}} = 1 \cdot 2,$$
$$K_{\ydiagram{2,2},\ydiagram{2,2}} K_{\ydiagram{2,2},\ydiagram{2,1,1}} = 1 \cdot 1, \qquad K_{\ydiagram{2,1,1},\ydiagram{2,2}} K_{\ydiagram{2,1,1},\ydiagram{2,1,1}} = 0 \cdot 1, \qquad K_{\ydiagram{1,1,1,1},\ydiagram{2,2}} K_{\ydiagram{1,1,1,1},\ydiagram{2,1,1}} = 0 \cdot 0.$$

Having reviewed the classical RSK bijection, the proof of the odd RSK Correspondence is simply a matter of keeping track of the signs associated to the combinatorial objects in question.

Proof of Theorem 2.2.16. In passing from a matrix A to the corresponding two-row array $\binom{\underline{u}}{\underline{v}} = \binom{u_1, u_2, \ldots}{v_1, v_2, \ldots}$, we have

$$\operatorname{sign}(\underline{u}) = 1, \qquad \operatorname{sign}(\underline{v}) = \operatorname{sign}(A).$$

As we construct the semistandard Young tableaux (P, Q) corresponding to A from the words $\underline{u}$ and $\underline{v}$, we will keep track of the signs of their row words at each step.

Each pair (u_j, v_j) describes an entry of the matrix A (we consider an entry equal to 2 as two entries equal to 1, and so forth). Let $A_1, A_2, \ldots$ be the sequence of matrices obtained by truncating $\underline{u}$ and $\underline{v}$. That is, under the first step of the RSK correspondence,

$$A_1 \leftrightarrow \begin{pmatrix} \underline{u}_1 \\ \underline{v}_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix},$$
$$A_2 \leftrightarrow \begin{pmatrix} \underline{u}_2 \\ \underline{v}_2 \end{pmatrix} = \begin{pmatrix} u_1, u_2 \\ v_1, v_2 \end{pmatrix},$$
$$A_3 \leftrightarrow \begin{pmatrix} \underline{u}_3 \\ \underline{v}_3 \end{pmatrix} = \begin{pmatrix} u_1, u_2, u_3 \\ v_1, v_2, v_3 \end{pmatrix},$$

and so forth. Let (P_k, Q_k) be the pair of semistandard Young tableaux corresponding to A_k, so that P_k (respectively Q_k) is obtained from P_{k-1} (respectively Q_{k-1}) by adding a box with label u_k (respectively v_k). Let $\lambda_{(k)} = \mathrm{sh}(P_k) = \mathrm{sh}(Q_k)$. Since the theorem is easily verified for one-box tableaux, it suffices to check that if $A_k \leftrightarrow (P_k, Q_k)$ and $\mathrm{sign}(A_k) = (-1)^{\binom{\lambda^T_{(k)}}{2} + N^<(P_k) + N^<(Q_k)}$, then passing to $A_{k+1}, P_{k+1}, Q_{k+1}, \lambda_{(k+1)}$ preserves this equality of signs. (The use of the notation $\lambda_{(k)}$ for a diagram rather than a row length should hopefully cause no confusion.)

To track the sign incurred in adding a box to P and to Q, note that both elementary Knuth transformations (K') and (K'') are transpositions (of letters in the words). It follows that

$$\mathrm{sign}(\underline{v}_k) = (-1)^{\mathrm{Kn}(\underline{v}_k) + N(\mathrm{sh}(Q_k)) + N^<(Q_k)},$$

where $\mathrm{Kn}(\underline{v}_k)$ is defined to be the number of elementary Knuth transformations needed to re-arrange $\underline{v}_k$ into $w_r(Q_k)$. Its residue modulo 2 is just the sign of the minimal length permutation sorting $\underline{v}_k$ into $w_r(Q_k)$. What needs to be shown, then, is that

$$\begin{aligned} (-1)^{\mathrm{Kn}(\underline{v}_k)} &= (-1)^{\binom{\lambda^T_{(k)}}{2} + N(\mathrm{sh}(P)) + N^<(P)} \\ &= (-1)^{(\lambda_{(k)})_2 + (\lambda_{(k)})_4 + (\lambda_{(k)})_6 + \ldots + N(\mathrm{sh}(P)) + N^<(P)}. \end{aligned} \tag{2.2.29}$$

Since this is clearly true for $k = 1$, we can prove this inductively by considering what happens when a new box is added.

Suppose when a new box with label v_{k+1} is added, it ends up in row $s+1$ ($s \geq 0$). In terms of tableaux, this means s boxes were bumped. Each time a bumping occurs in row

j, the number of elementary Knuth transformations which take place is $(\lambda_{(k)})_j - 1$. So

$$\mathrm{Kn}(\underline{v}_{k+1}) = \mathrm{Kn}(\underline{v}_k) + \sum_{j=1}^{s}((\lambda_{(k)})_j - 1).$$

And by definition of s, the sign $(-1)^{\lambda_2+\lambda_4+\lambda_6+\dots}$ changes by $(-1)^s$ in passing from $\lambda_{(k)}$ to $\lambda_{(k+1)}$. Finally,

$$(-1)^{N(\mathrm{sh}(P_{k+1}))+N^{<}(P_{k+1})} = (-1)^{(\lambda_{(k)})_1+\dots+(\lambda_{(k)})_s+N(\mathrm{sh}(P_k))+N^{<}(P_k)},$$

since u_k is greater than any label it passes through in sorting $w_r(P_k)u_{k+1}$ to $w_r(P_{k+1})$. We see that the sign changes in the factors of equation (2.2.29) cancel, so the sign equality is preserved under the addition of a new box. The completes the proof of the theorem. $\square$

Example 2.2.26. We return to Example 2.2.24. Next to each combinatorial object (including the semistandard Young tableau shapes), we put the associated sign.

$$\begin{array}{llll}
+\begin{pmatrix}1&0\\1&0\\0&1\end{pmatrix} & +\,\ydiagram{3} & +\begin{ytableau}1&1&2\end{ytableau}, & +\begin{ytableau}1&2&3\end{ytableau}, \\
-\begin{pmatrix}1&0\\0&1\\1&0\end{pmatrix} & -\,\ydiagram{2,1} & +\begin{ytableau}1&1\\2\end{ytableau}, & +\begin{ytableau}1&2\\3\end{ytableau}, \\
+\begin{pmatrix}0&1\\1&0\\1&0\end{pmatrix} & -\,\ydiagram{2,1} & +\begin{ytableau}1&1\\2\end{ytableau}, & -\begin{ytableau}1&3\\2\end{ytableau}.
\end{array}$$

Example 2.2.27. We return to Example 2.2.25.

$$
\begin{array}{llll}
+\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} & + \begin{array}{|c|c|c|c|}\hline & & & \\ \hline \end{array} & + \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline \end{array}, & + \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline \end{array}, \\
-\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} & - \begin{array}{|c|c|c|}\hline & & \\ \hline & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} & + \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array}, & + \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array}, \\
+\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} & - \begin{array}{|c|c|c|}\hline & & \\ \hline & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} & - \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array}, & + \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array}, \\
+\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix} & + \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} & + \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array}, & + \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}.
\end{array}
$$

2.3 Questions: geometry, categorification, and structure

In this chapter we have presented the basic theory of odd symmetric functions. A few problems are immediately suggested:

- Find odd analogues of more even case objects—e.g. Hall-Littlewood and Macdonald polynomials, vertex operators on Λ, quasideterminantal formulae, and so forth;
- Study Hopf-theoretic properties of $O\Lambda$ such as Jordan blocks and eigenvalues for convolution-powers of the identity;
- Study Λ^q for q a primitive n-th root of unity, $n > 2$;
- Understand why $O\Lambda$ has no primitives in odd degrees greater than 1.

There are also larger questions. The ordinary (even case) symmetric functions and variants thereof are controlled by higher structures in various ways. The Hopf algebra Λ is categorified by the complex representation theory of the symmetric groups, finite variable quotients of Λ are character rings for polynomial representations of $GL_n(\mathbb{C})$, quotients of Λ arise as the cohomology algebras of Grassmannians, the coinvariant algebra $\mathrm{Pol}_n/\Lambda_n^+$ is the cohomology algebra of the full flag variety (and other quotients give the cohomology of Springer fibers), various related Hecke algebras arise as convolution algebras for different cohomology theories (singular, Borel-Moore, complex K-theory, and equivariant variants thereof)—the list goes on.

Are there odd analogues of these? This question strikes deeper than the more naïve questions above. Here are some reasons why:

- The bilinear form on Λ is positive definite, symmetric, and bilinear. On $O\Lambda$ the form is still nondegenerate, symmetric, and bilinear, but it is indefinite. Thus if it is to arise as a hom-form coming from categorification—and the combinatorics strongly suggest this—then this categorification cannot be abelian. Whatever category provides the correct notion of "odd symmetric group," it must derived, dg-, triangulated, or similar. This is in stark contrast to complex representation category of S_n, which is finitary, Frobenius, semisimple, and abelian.

- As a baby step towards categorifying $O\Lambda$, one can try and give a graded categorification of $N\Lambda^q$ for generic q, where q is the decategorification of a grading shift. A weaker filtered categorification has been given by Thibon and others [Thi01; TU96] using a filtration on cyclic modules for 0-Hecke algebras, but something genuinely graded would be preferable.

- Odd analogues of the Grassmannian cohomology ring can be defined by quotients analogous to those in the even case (see the discussion of cyclotomic quotients in Chapter 3). The algebras are not supercommutative, but by the odd Littlewood-Richardson rule, their combinatorics do suggest an "odd Schubert calculus." What sort of geometric object has this "odd cohomology," and what is its intersection theory? Mere supergeometry is not noncommutative enough.

- The odd analogues of the coinvariant algebra and Springer fiber cohomology introduced by Lauda and Russell [LR12] are not even algebras or skew polynomial bimodules. In what sort of algebraic or structural context do these objects belong?

Again, the list can go on, but perhaps it is better to stop here.

Chapter 3

Odd nilHecke algebras

The results of this chapter, many of which are joint work with Mikhail Khovanov and Aaron Lauda, are taken from [EKL12; Ell12].

3.1 Odd symmetric polynomials

3.1.1 Defining the odd symmetric polynomials

We define the ring of *skew polynomials* to be

$$\mathrm{Pol}_a^{-1} = \mathbb{Z}\langle x_1, \dots, x_a \rangle / \langle x_i x_j + x_j x_i = 0 \text{ for } i \neq j \rangle. \tag{3.1.1}$$

We let the symmetric group S_a act on the degree k part of Pol_a^{-1} as the tensor product of the permutation representation and the k-th tensor power of the sign representation. That is, for $1 \leq j \leq a-1$, the transposition $s_j \in S_a$ acts as the ring endomorphism

$$s_i(x_j) = \begin{cases} -x_{i+1} & \text{if } j = i, \\ -x_i & \text{if } j = i+1, \\ -x_j & \text{otherwise.} \end{cases} \tag{3.1.2}$$

The *odd divided difference operators* are the linear operators ∂_i $(1 \leq i \leq a-1)$ on the free associative algebra $\mathbb{Z}\langle x_1, \ldots, x_a \rangle$ defined by

$$\partial_i(1) = 0,$$

$$\partial_i(x_j) = \begin{cases} 1 & \text{if } j = i, i+1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1.3}$$

and the Leibniz rule

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g) \text{ for all } f, g \in \mathbb{Z}\langle x_1, \ldots, x_a \rangle. \tag{3.1.4}$$

It is easy to check from the definition of ∂_i that for all i,

$$\partial_i(x_j x_k + x_k x_j) = 0 \text{ for } j \neq k, \tag{3.1.5}$$

so ∂_i descends to an operator on Pol_a^{-1}.

Note that Pol_a^{-1} is left and right Noetherian and has no zero divisors, but does not satisfy the unique factorization property if $a \geq 2$:

$$x_1^2 + x_2^2 = (x_1 - x_2)^2 = (x_1 + x_2)^2.$$

Since $x_i^2 \neq 0$, $opol_a$ is not super-commutative.

The following basic formulae in Pol_a^{-1} can be derived from the above.

$$\begin{aligned} &\partial_i(x_i - x_{i+1}) = 0, \qquad \partial_i(x_i x_{i+1}) = 0, \\ &\partial_i(x_i^{2m} + x_{i+1}^{2m}) = 0, \qquad \partial_i(x_i^m x_{i+1}^m) = 0, \\ &\partial_i(x_i^m) = \sum_{j=0}^{m-1} (-1)^j x_{i+1}^j x_i^{m-1-j}, \\ &\partial_i(x_{i+1}^m) = \sum_{j=0}^{m-1} (-1)^j x_i^j x_{i+1}^{m-1-j}. \end{aligned} \tag{3.1.6}$$

Proposition 3.1.1. Considering ∂_i and (multiplication by) x_j as operators on Pol_a^{-1}, the following relations hold in $\mathrm{End}(\mathrm{Pol}_a^{-1})$:

$$\partial_i^2 = 0, \quad \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}, \tag{3.1.7}$$

$$x_i \partial_i + \partial_i x_{i+1} = 1, \quad \partial_i x_i + x_{i+1} \partial_i = 1, \tag{3.1.8}$$

$$x_i x_j + x_j x_i = 0 \quad (i \neq j), \quad \partial_i \partial_j + \partial_j \partial_i = 0 \quad (|i-j| > 1), \tag{3.1.9}$$

$$x_i \partial_j + \partial_j x_i = 0 \quad (i \neq j, j+1). \tag{3.1.10}$$

Proof. We first prove that $\partial_i^2(f) = 0$ for any $f \in \mathrm{Pol}_a^{-1}$. We can reduce to the case $i = 1$ and f being a monomial in x_1, x_2, and then proceed by induction on the degree of f. When the degree is zero, $f = 1$ and $\partial_1^2(1) = 0$. Assume $\partial_1^2(f) = 0$ for any monomial of degree m. Then

$$\partial_1^2(x_1 f) = \partial_1(f - x_2\partial_1(f)) = \partial_1(f) - \partial_1(x_2)\partial_1(f) + x_1\partial_1^2(f) = \partial_1(f) - \partial_1(f) = 0,$$
$$\partial_1^2(x_2 f) = \partial_1(f - x_1\partial_1(f)) = \partial_1(f) - \partial_1(x_1)\partial_1(f) + x_2\partial_1^2(f) = 0,$$

which takes care of the inductive step. Next, we verify relations of the second type from (3.1.7). It suffices to assume that $i = 1$ and the action is on a monomial in x_1, x_2, x_3. We proceed by induction on the degree of the monomial. When $f = 1$, both sides act by 0. Assuming that

$$\partial_1\partial_2\partial_1(f) = \partial_2\partial_1\partial_2(f)$$

we compute

$$\begin{aligned}
\partial_1\partial_2\partial_1(x_1 f) &= \partial_1\partial_2(f - x_2(\partial_1(f))) = \partial_1(\partial_2(f) - \partial_1(f) + x_3(\partial_2\partial_1(f))) \\
&= \partial_1\partial_2(f) - \partial_1^2(f) - x_3(\partial_1\partial_2\partial_1(f)) = \partial_1\partial_2(f) - x_3(\partial_1\partial_2\partial_1(f)), \\
\partial_2\partial_1\partial_2(x_1 f) &= \partial_2\partial_1(-x_2\partial_2(f)) = \partial_2(-\partial_2(f) + x_2\partial_1\partial_2(f)) \\
&= \partial_1\partial_2(f) - x_3(\partial_2\partial_1\partial_2(f)),
\end{aligned}$$

implying $\partial_1\partial_2\partial_1(x_1 f) = \partial_2\partial_1\partial_2(x_1 f)$. Likewise,

$$\begin{aligned}
\partial_1\partial_2\partial_1(x_2 f) &= \partial_1\partial_2(f - x_1(\partial_1(f))) = \partial_1(\partial_2(f) + x_1(\partial_2\partial_1(f))) \\
&= \partial_1\partial_2(f) + \partial_2\partial_1(f) - x_2(\partial_1\partial_2\partial_1(f)), \\
\partial_2\partial_1\partial_2(x_2 f) &= \partial_2\partial_1(f - x_3\partial_2(f)) = \partial_2(\partial_1(f) + x_3\partial_1\partial_2(f)) \\
&= \partial_1\partial_2(f) + \partial_2\partial_1(f) - x_2\partial_2\partial_1\partial_2(f),
\end{aligned}$$

proving the inductive step in this case. Checking that the two actions are the same on $x_3 f$ is equally simple.

Relations (3.1.8) follow from the Leibniz rule (3.1.4):

$$\begin{aligned}
\partial_i(x_{i+1}f) &= \partial(x_{i+1})f - x_i\partial_i(f) = f - x_i\partial_i(f), \\
\partial_i(x_i f) &= f - x_{i+1}\partial_i(f).
\end{aligned}$$

The first type of relation in (3.1.9) consists of defining relations in Pol_a^{-1}, and the second type in (3.1.9) can be checked by applying $\partial_i\partial_j + \partial_j\partial_i$ to monomials in $x_1, \ldots, x_a$ to get 0. The relation (3.1.10) is again a special case of (3.1.4). $\square$

> ***From now on, we use the following gradings:***
> The degree of x_i is 2, so the degree of ∂_i is -2. While this contrasts with the previous chapter, it is essential for getting the gradings right in categorification. The choice is motivated by thinking of $x_1 + \ldots + x_a$ as the first Chern class of a certain vector bundle.

Since $\partial_i^2 = 0$, we can consider Pol_a^{-1} as a chain complex (taking the homological grading to be one-half of the grading $\deg(x_i) = 2$, so $(\mathrm{Pol}_a^{-1})_{2k}$ sits in homological degree k). The multiplication operators

$$\begin{gathered}\psi_k : (\mathrm{Pol}_a^{-1})_k \to (\mathrm{Pol}_a^{-1})_{k+1} \\ \psi_k = \begin{cases} x_i & k \text{ is even} \\ x_{i+1} & k \text{ is odd} \end{cases}\end{gathered} \tag{3.1.11}$$

give a chain homotopy between the identity and the zero maps, so the complex $((\mathrm{Pol}_a^{-1})_k, \partial_i)$ is contractible for each i, a. In particular, $\ker(\partial_i) = \mathrm{im}\,(\partial_i)$ for each i. We define the ring of *odd symmetric polynomials* to be the subring

$$O\Lambda_a = \bigcap_{i=1}^{a-1} \ker(\partial_i) = \bigcap_{i=1}^{a-1} \mathrm{im}\,(\partial_i) \tag{3.1.12}$$

of Pol_a^{-1}. Observe that

$$\begin{gathered}\mathrm{Pol}_a^{-1} \otimes_{\mathbb{Z}} (\mathbb{Z}/2) \cong \mathrm{Pol}_a \otimes_{\mathbb{Z}} (\mathbb{Z}/2) \\ O\Lambda_a \otimes_{\mathbb{Z}} (\mathbb{Z}/2) \cong \Lambda_a \otimes_{\mathbb{Z}} (\mathbb{Z}/2),\end{gathered} \tag{3.1.13}$$

the usual (commutative) rings of polynomials and symmetric polynomials in a variables over $\mathbb{Z}/2$. In particular, both Pol_a^{-1} and $O\Lambda_a$ are free abelian groups whose ranks in each degree are bounded below by those of Pol_a and Λ_a, since $\mathrm{rk}_q(\mathrm{Pol}_a) = (\dim_q)_{\mathbb{Z}/2}(\mathrm{Pol}_a)$ and

$\mathrm{rk}_q(\Lambda_a) = (\dim_q)_{\mathbb{Z}/2}(\Lambda_a)$. Here, for a graded abelian group V of finite rank in each degree and a graded vector space W over a field $\mathbb{F}$ of finite dimension in each degree,

$$\begin{aligned} \mathrm{rk}_q(V) &= \sum_{i\in\mathbb{Z}} \mathrm{rk}(V_i) q^i, \\ (\dim_q)_{\mathbb{F}}(W) &= \sum_{i\in\mathbb{Z}} \dim_{\mathbb{F}}(W_i) q^i. \end{aligned} \tag{3.1.14}$$

Both are power series in $q^{\pm 1}$. It is clear that Pol_a and Pol_a^{-1} have the same graded rank.

> ***Warning:***
>
> This definition of the odd symmetric polynomials almost, but does not quite agree with the image of the map $\Lambda^{-1} \to Q\Lambda^{-1} \to \mathrm{Pol}_a^{-1}$. The two subalgebras are carried to each other by the map involution of Pol_a^{-1} taking $x_i \mapsto \widetilde{x}_i$; this is clear from Lemma 3.1.3, below. Our differing notation for elementary functions in these two contexts is meant to serve as a reminder of this differnece.

3.1.2 Odd elementary symmetric polynomials and the size of Pol_a^{-1}

The graded rank of Λ_a is

$$\begin{aligned} \mathrm{rk}_q(\Lambda_a) &= \prod_{i=1}^{a} \frac{1}{1-q^{2i}} \\ &= \frac{1}{(1-q^2)^a} \prod_{i=1}^{a} \left(q^{-i+1} \frac{q-q^{-1}}{q^a - q^{-a}} \right) \\ &= q^{-\binom{a}{2}} \frac{1}{[a]!} \frac{1}{(1-q^2)^a}, \end{aligned} \tag{3.1.15}$$

an element of $\mathbb{Z}_{\geq 0}[\![q^2]\!]$. This equals

$$\mathrm{rk}_q(\Lambda_a) = \frac{\mathrm{rk}_q(\mathrm{Pol}_a)}{\sum_{\sigma\in S_a} q^{\ell(\sigma)}}, \tag{3.1.16}$$

where ℓ denotes the Coxeter length function on S_a; this makes one think of "taking a quotient of the ring of all polynomials by the symmetric group."

Proposition 3.1.2. The rings of symmetric and odd symmetric polynomials have the same graded ranks:

$$\mathrm{rk}_q(O\Lambda_a) = \mathrm{rk}_q(\Lambda_a) = q^{-\binom{a}{2}} \frac{1}{[a]!} \frac{1}{(1-q^2)^a}. \tag{3.1.17}$$

To prove this proposition, we will have to develop odd analogues of the elementary symmetric polynomials.

By analogy with the even case, we introduce the odd elementary symmetric polynomials

$$\varepsilon_k(x_1,\ldots,x_a) = \sum_{1\le i_1<\cdots<i_k\le a} \widetilde{x}_{i_1}\cdots\widetilde{x}_{i_k}, \qquad \text{where } \widetilde{x}_i = (-1)^{i-1}x_i, \tag{3.1.18}$$

for $1 \le k \le a$. If $k = 0$ define $\varepsilon_k = 1$, and if $k < 0$ or $k > a$ define $\varepsilon_k = 0$. If we want to emphasize the number of variables in shorthand, we will write $\varepsilon_k^{(a)}$ for $\varepsilon_k(x_1,\ldots,x_a)$. Modulo 2, the elementary symmetric and odd elementary symmetric polynomials are the same. But with signs, the relations among the odd ε_k are more complicated than mere commutativity. The following lemma will give us enough relations to write down a presentation of $O\Lambda_a$.

Lemma 3.1.3. The polynomial $\varepsilon_k(x_1,\ldots,x_a)$ is odd symmetric for $1 \le k \le a$. Furthermore, the following relations hold in the ring $O\Lambda_a$:

$$\begin{aligned}
&\varepsilon_i\varepsilon_{2m-i} = \varepsilon_{2m-i}\varepsilon_i \qquad (1 \le i, 2m-i \le a),\\
&\varepsilon_i\varepsilon_{2m+1-i} + (-1)^i\varepsilon_{2m+1-i}\varepsilon_i = (-1)^i\varepsilon_{i+1}\varepsilon_{2m-i} + \varepsilon_{2m-i}\varepsilon_{i+1} \qquad (1 \le i, 2m-i \le a-1),\\
&\varepsilon_1\varepsilon_{2m} + \varepsilon_{2m}\varepsilon_1 = 2\varepsilon_{2m+1} \qquad (1 < 2m \le a-1).
\end{aligned} \tag{3.1.19}$$

Note that the third is the $i = 0$ case of the second.

By the first relation, odd subscripts commute with odd subscripts and even subscripts commute with even subscripts. These relations are enough to reduce any word $\varepsilon_{i_1}\cdots\varepsilon_{i_r}$ to a $\mathbb{Z}$-linear combination of terms of the form $\varepsilon_{j_1}\cdots\varepsilon_{j_s}$ with $j_1 \ge \cdots \ge j_s$; hence the rank of $O\Lambda_a$ in each degree is bounded above by that of Λ_a in the same rank (cf. the proof of Proposition 3.1.2 below).

Proof. The relations are all true in degrees $0, 1$. The third relation is the $i = 0$ case of the second. In $a = 1$ variable, all the relations are clear. We now prove the first and second relations simultaneously by induction on the number of variables a. The equation

$$\varepsilon_m^{(a)} = \varepsilon_m^{(a-1)} + \varepsilon_{m-1}^{(a-1)}\widetilde{x}_a \tag{3.1.20}$$

allows us to reduce calculations to fewer variables. Now we compute (writing just ε_j for $\varepsilon_j^{(a-1)}$):

$$\begin{aligned}\varepsilon_i^{(a)}\varepsilon_{2m-i}^{(a)} - \varepsilon_{2m-i}^{(a)}\varepsilon_i^{(a)} &= (\varepsilon_i + \varepsilon_{i-1}\widetilde{x}_a)(\varepsilon_{2m-i} + \varepsilon_{2m-i-1}\widetilde{x}_a) - (\varepsilon_{2m-i} + \varepsilon_{2m-i-1}\widetilde{x}_a)(\varepsilon_i + \varepsilon_{i-1}\widetilde{x}_a)\\ &= (\varepsilon_i\varepsilon_{2m-i} - \varepsilon_{2m-i}\varepsilon_i) + (-1)^{i-1}(\varepsilon_{i-1}\varepsilon_{2m-i-1} - \varepsilon_{2m-i-1}\varepsilon_{i-1})\widetilde{x}_a^2\\ &\qquad + \left((-1)^i\varepsilon_{i-1}\varepsilon_{2m-i} - \varepsilon_{2m-i}\varepsilon_{i-1} + \varepsilon_i\varepsilon_{2m-i-1} - (-1)^i\varepsilon_{2m-i-1}\varepsilon_i\right)\widetilde{x}_a\\ &= 0.\end{aligned} \tag{3.1.21}$$

The second equality was grouping terms into powers of $\widetilde{x}_a$; the third equality used the induction hypothesis on the coefficients of 1 and of $\widetilde{x}_a^2$ (first relation) and on the coefficient of $\widetilde{x}_a$ (second relation). The second relation is proved similarly. □

Remark 3.1.4. The relations (3.1.19) allow one to sort a product $\varepsilon_{i_1}\cdots\varepsilon_{i_k}$ into non-increasing order of subscripts (up to sign), modulo a set of terms which are lexicographically higher and in $2\mathbb{Z}\cdot O\Lambda_a$. This follows from a consequence of the odd degree relation: suppose $k < \ell$ and $k+\ell$ is odd. Then

$$\begin{aligned}\varepsilon_k\varepsilon_\ell &= \varepsilon_\ell\varepsilon_k + 2\sum_{i=1}^{k}(-1)^{\binom{i}{2}}\varepsilon_{\ell+i}\varepsilon_{k-i} && \text{if } k \text{ is even},\\ \varepsilon_k\varepsilon_\ell &= -\varepsilon_\ell\varepsilon_k + 2\sum_{i=1}^{k}(-1)^{\binom{i-1}{2}}\varepsilon_{\ell+i}\varepsilon_{k-i} && \text{if } k \text{ is odd}.\end{aligned} \tag{3.1.22}$$

Remark 3.1.5. Of course, the relations (3.1.19) are identical to (2.2.4),(2.2.5). Though note that in this finite-variables context, $\varepsilon_k = 0$ if $k > a$.

Proof of Proposition 3.1.2. Let $O\Lambda_a^{\mathrm{elem}}$ be the subring of $O\Lambda_a$ generated by the odd elementary symmetric polynomials $\varepsilon_k(x_1, \ldots, x_a)$. For a partition $\alpha = (\alpha_1, \ldots, \alpha_m)$ of a written in non-increasing order, let

$$\varepsilon_\alpha = \varepsilon_{\alpha_1}\cdots\varepsilon_{\alpha_m}. \tag{3.1.23}$$

Lemma 3.1.3 and the remark which follows it imply that these products span $O\Lambda_a^{\mathrm{elem}}$. Hence the rank of each graded piece of $O\Lambda_a^{\mathrm{elem}}$ is bounded above by that of the corresponding graded piece of Λ_a. Conversely, we have an isomorphism

$$\Lambda_a \otimes_{\mathbb{Z}} (\mathbb{Z}/2) \cong O\Lambda_a \otimes_{\mathbb{Z}} (\mathbb{Z}/2) \cong O\Lambda_a^{\mathrm{elem}} \otimes_{\mathbb{Z}} (\mathbb{Z}/2). \tag{3.1.24}$$

The identification of the first and third follows because the generators and relations of $O\Lambda_a^{\mathrm{elem}}$ and Λ_a coincide mod 2; the identification of the first and second follows because mod 2, the action of the divided difference operators on Pol_a and the action of the odd divided difference operators on Pol_a^{-1} coincide. The graded rank of Λ_a and the graded dimension of $\Lambda_a \otimes_{\mathbb{Z}} (\mathbb{Z}/2)$ are equal. This bounds the graded rank of $O\Lambda_a^{\mathrm{elem}}$ below by that of Λ_a: indeed, dividing any linear relation between the ε_α in $O\Lambda_a^{\mathrm{elem}}$ by a high enough power of 2, we would obtain a linear relation between their reductions mod 2, a contradiction. Thus $\mathrm{rk}_q(\Lambda_a) = \mathrm{rk}_q(O\Lambda_a^{\mathrm{elem}})$.

To conclude that $O\Lambda_a = O\Lambda_a^{\mathrm{elem}}$, we prove that both have free abelian complements in Pol_a^{-1}. For $O\Lambda_a$, this is because if there were no free complement, some free direct summand (as a $\mathbb{Z}$-submodule) would be wholly divisible by an integer $d > 1$. But then we could divide generators of this summand by d. The result would still be in the kernel of all the operators ∂_i, a contradiction. As for $O\Lambda_a^{\mathrm{elem}}$, one checks that with respect to a lexicographic order on monomials, the highest order term of ε_α is $\underline{x}^\alpha = x_1^{\alpha_1} \cdots x_r^{\alpha_m}$ with coefficient 1. Now since $O\Lambda_a^{\mathrm{elem}} \subseteq O\Lambda_a$ and both have free complements, the graded dimensions over $\mathbb{Z}/2$ of their reductions mod 2 coincide if and only if they are equal. As both rings have mod 2 reduction isomorphic to Λ_a, these graded dimensions do coincide. So $O\Lambda_a^{\mathrm{elem}} = O\Lambda_a$, and we have established the formula

$$\mathrm{rk}_q(O\Lambda_a) = q^{-\binom{a}{2}} \frac{1}{[a]!} \frac{1}{(1-q^2)^a}. \tag{3.1.25}$$

$\square$

Corollary 3.1.6. The graded superalgebra $O\Lambda$, as defined in the previous chapter, is isomorphic to the limit of the graded superalgebras $O\Lambda_n$. That is,

$$O\Lambda = \varprojlim_n O\Lambda_n, \tag{3.1.26}$$

The following lemma is useful for passing between the rings $O\Lambda_a$ and $O\Lambda_{a-1}$.

Lemma 3.1.7. For any $k \geq 0$,

$$\begin{aligned}\varepsilon_k^{(a-1)} &= \varepsilon_k^{(a)} - \varepsilon_{k-1}^{(a)}\widetilde{x}_a + \varepsilon_{k-2}^{(a)}\widetilde{x}_a^2 - \ldots + (-1)^k\widetilde{x}_a^k \\ &= \sum_{j=0}^{k}(-1)^j\varepsilon_{k-j}^{(a)}\widetilde{x}_a^j.\end{aligned} \tag{3.1.27}$$

Proof. The polynomial $\varepsilon_k^{(a)}$ consists of terms without $\widetilde{x}_a$ and terms with $\widetilde{x}_a$. The former add up precisely to $\varepsilon_k^{(a-1)}$. The latter all appear as terms in $\varepsilon_{k-1}^{(a)}\widetilde{x}_a$; when we subtract off $\varepsilon_{k-1}^{(a)}\widetilde{x}_a$, the extra terms subtracted are precisely those which already had a factor of $\widetilde{x}_a$ in $\varepsilon_{k-1}^{(a)}$. The extra terms are now those with a factor of $\widetilde{x}_a^2$, and they appear in $\varepsilon_{k-2}^{(a)}\widetilde{x}_a^2$. Continuing in this fashion (inclusion-exclusion), the formula follows. □

By a slight abuse of notation, if $R \subseteq S$ is a subring and $s \in S$, we write $R[s]$ for the subring of S generated by R and s.

Corollary 3.1.8. Inside Pol_a^{-1}, $O\Lambda_a[x_a] = O\Lambda_{a-1}[x_a]$.

Proof. "$\subseteq$": Let $f \in O\Lambda_a$. Using skew-commutativity, we can move all factors of x_a in any term of f all the way to the right. Collecting powers of x_a, we see each coefficient of a given power of x_a is an element of $O\Lambda_{a-1}$, so $O\Lambda_a \subseteq O\Lambda_{a-1}[x_a]$. The converse "$\supseteq$" follows from the previous lemma. □

3.1.3 Odd complete symmetric polynomials

Definition 3.1.9. For $k \geq 1$, the k-th *odd complete symmetric polynomial* is defined to be

$$h_k(x_1, \ldots, x_a) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq a} \widetilde{x}_{i_1} \cdots \widetilde{x}_{i_k}. \tag{3.1.28}$$

Also define $h_0 = 1$ and $h_k = 0$ for $k < 0$.

Lemma 3.1.10. The polynomials ε_k and h_k in Pol_a^{-1} satisfy

$$\sum_{k=0}^{m}(-1)^{\binom{k+1}{2}}\varepsilon_k h_{m-k} = 0 \tag{3.1.29}$$

for all $m \geq 1$.

Proof. We proceed by induction on the number of variables a, the case $a = 1$ being clear. Let ε_k, h_k denote the odd elementary and odd complete polynomials in a variables, so that

$$\varepsilon_k^{(a+1)} = \varepsilon_k + \varepsilon_{k-1}\widetilde{x}_{a+1}, \qquad h_k^{(a+1)} = \sum_{j=0}^{k} h_{k-j}\widetilde{x}_{a+1}^j.$$

Plugging these expressions into the left-hand side of equation (3.1.29) in $a+1$ variables,

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^{\binom{k+1}{2}}\varepsilon_k^{(a+1)}h_{n-k}^{(a+1)} &= \sum_{k=0}^{n}(-1)^{\binom{k+1}{2}}(\varepsilon_k + \varepsilon_{k-1}\widetilde{x}_{a+1})\left(\sum_{j=0}^{n-k} h_{n-k-j}\widetilde{x}_{a+1}^j\right)\\
&= \sum_{k=0}^{n}\sum_{j=0}^{n-k}(-1)^{\binom{k+1}{2}}\left(\varepsilon_k h_{n-k-j} + (-1)^{n-k-j}\varepsilon_{k-1}h_{n-k-j}\widetilde{x}_{a+1}\right)\widetilde{x}_{a+1}^j\\
&= \sum_{j=0}^{n}\sum_{k=0}^{n-j}(-1)^{\binom{k+1}{2}}\varepsilon_k h_{n-k-j}\widetilde{x}_{a+1}^j\\
&\quad + \sum_{j=1}^{n+1}\sum_{k=0}^{n-j+1}(-1)^{\binom{k+1}{2}+n-k+j-1}\varepsilon_{k-1}h_{n-k-j+1}\widetilde{x}_{a+1}^j.
\end{aligned}$$

In the last equality, the order of summation was reversed and the second term was re-indexed, $j \mapsto j-1$. Consider the last line: the inner sum of the first term is zero unless $j = n, k = 0$ (induction hypothesis). Also, the second term is 0 when $k = 0$. Removing these vanishing terms, removing boundary terms from the summations, re-combining the two summation terms, and re-indexing $k \mapsto k+1$, this equals

$$\ldots = \widetilde{x}_{a+1}^n + \sum_{j=1}^{n+1}(-1)^{n+j-1}\left(\sum_{k=0}^{n-j}(-1)^{\binom{k+1}{2}}\varepsilon_k h_{n-k-j}\right)\widetilde{x}_{a+1}^j.$$

The inner sum here is zero unless $j = n, k = 0$ (induction hypothesis), in which case it cancels the $\widetilde{x}_{a+1}^n$. So the entire expression equals zero, and we are done. □

Equation (3.1.29) can be used inductively to solve for each h_m as a polynomial in the various ε_k, so each h_m is indeed odd symmetric.

Lemma 3.1.11. The polynomials h_k satisfy the same relations as the ε_k in the ring $O\Lambda_a$:

$$\begin{aligned}
&h_i h_{2m-i} = h_{2m-i}h_i \qquad (1 \le i, 2m-i \le a),\\
&h_i h_{2m+1-i} + (-1)^i h_{2m+1-i}h_i = (-1)^i h_{i+1}h_{2m-i} + h_{2m-i}h_{i+1} \qquad (1 \le i, 2m-i \le a-1),\\
&h_1 h_{2m} + h_{2m}h_1 = 2h_{2m+1} \qquad (1 < 2m \le a-1).
\end{aligned} \tag{3.1.30}$$

Furthermore, we have the following mixed relations:

$$\begin{aligned}
&\varepsilon_i h_{2m-i} = h_{2m-i}\varepsilon_i \qquad (1 \leq i, 2m-i \leq a),\\
&\varepsilon_i h_{2m+1-i} + (-1)^i h_{2m+1-i}\varepsilon_i = (-1)^i \varepsilon_{i+1} h_{2m-i} + h_{2m-i}\varepsilon_{i+1} \qquad (1 \leq i, 2m-i \leq a-1).
\end{aligned} \tag{3.1.31}$$

Proof. This follows from the isomorphism with a quotient $O\Lambda$ as defined in the previous chapter, but we give a direct proof in terms of skew polynomials. The proofs of all these relations are similar to the proof of Lemma 3.1.3, using equation (3.1.20) and its complete polynomial analogue

$$h_m^{(a)} = \sum_{j=0}^{m} h_{k-j}^{(a-1)} \widetilde{x}_a^j. \tag{3.1.32}$$

□

As in the case of elementary functions, define $h_\alpha = h_{\alpha_1} \cdots h_{\alpha_m}$ for a partition $\alpha = (\alpha_1, \ldots, \alpha_m)$.

In the even case, monomial symmetric functions are defined by

$$m_\lambda = \sum_\alpha \underline{x}^\alpha \qquad \text{(sum over all distinct permutations } \alpha \text{ of } \lambda) \tag{3.1.33}$$

for a partition α. However, for certain λ, no such analogous sum yields an odd symmetric polynomial. An equivalent definition of these functions in the even setting is that they form the basis dual to the basis $\{h_\lambda\}_\lambda$ with respect to a standard bilinear form; hence our definition of m_λ in the previous chapter as a dual basis.

3.2 The odd nilHecke algebra

By analogy with the even case [Man01], we define the odd nilHecke ring ONH_a to be the graded unital associative ring generated by elements $x_1, \ldots, x_a$ of degree 2 and elements $\partial_1, \ldots, \partial_{a-1}$ of degree -2, subject to the relations (3.1.7), (3.1.8), (3.1.9), (3.1.10), which we repeat here:

$$\begin{aligned}
&\partial_i^2 = 0, \quad \partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1},\\
&x_i\partial_i + \partial_i x_{i+1} = 1, \quad \partial_i x_i + x_{i+1}\partial_i = 1,\\
&x_i x_j + x_j x_i = 0 \quad (i \neq j), \quad \partial_i\partial_j + \partial_j\partial_i = 0 \quad (|i-j| > 1),\\
&x_i\partial_j + \partial_j x_i = 0 \quad (i \neq j, j+1).
\end{aligned}$$

The odd nilHecke algebra is a close relative of spin Hecke algebras originally introduced by Weiqiang Wang and collaborators in connection with representation theory of the spin symmetric group [Wan09; KW08a; KW08b; KW09].

We define the odd nilCoxeter ring ONC_a to be the graded subring generated by the ∂_i's (this is the LOT ring of [LOT08]). As a consequence of these relations, ONH_a and ONC_a have natural representations on Pol_a^{-1}. The $\mathbb{Z}$-grading on ONH_a induces a $\mathbb{Z}/2$-grading given by dividing the $\mathbb{Z}$-grading by 2 and then reducing mod 2. For $f \in \mathrm{ONH}_a$ we write $\deg_s(f)$ for the super degree of f.

For each $w \in S_a$, choose a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ in terms of simple transpositions $s_i = (i \quad i+1)$. Define

$$\partial_w = \partial_{i_1} \cdots \partial_{i_\ell}. \tag{3.2.1}$$

For $w = w_0$ we make a particular choice of word and re-name the operator,

$$D_a = \partial_{w_0} = \partial_1(\partial_2\partial_1)\cdots(\partial_{a-1}\cdots\partial_1).$$

Since the ∂_i satisfy a signed version of the relations of the simple transpositions s_i, the definition of ∂_w is almost independent of choice of reduced expression for w—the only ambiguity is an overall sign. For $w, w' \in S_a$, the formula

$$\partial_w\partial_{w'} = \begin{cases} \pm\partial_{ww'} & \text{if } \ell(ww') = \ell(w) + \ell(w'),\\ 0 & \text{otherwise} \end{cases} \tag{3.2.2}$$

is proven just as in the even case [Man01].

When no confusion will result, if $A = (r_1, \ldots, r_a)$ is an a-tuple of integers we define

$$\underline{x}^A = x_1^{r_1} \cdots x_a^{r_a}. \tag{3.2.3}$$

Now it is clear that the sets

$$\{\underline{x}^A \partial_w\}_{w \in S_a, A \in \mathbb{Z}^n_{\geq 0}} \text{ and } \{\partial_w \underline{x}^A\}_{w \in S_a, A \in \mathbb{Z}^n_{\geq 0}} \tag{3.2.4}$$

generate ONH_a (for us, $0 \in \mathbb{Z}_{\geq 0}$). In fact, they are linearly independent as well. To prove this, we will introduce odd Schubert polynomials.

Let $\delta_a = (a-1, a-2, \ldots, 1, 0)$. In what follows we will make repeated use of the monomial

$$\underline{x}^{\delta_a} = x_1^{a-1} x_2^{a-2} \cdots x_{a-1}^1 x_a^0. \tag{3.2.5}$$

For $w \in S_a$, define the corresponding *odd Schubert polynomial* $\mathfrak{s}_w \in \mathrm{Pol}_a^{-1}$ by

$$\mathfrak{s}_w(x_1, \ldots, x_a) = \partial_{w^{-1}w_0}(\underline{x}^{\delta_a}), \tag{3.2.6}$$

where w_0 is the longest element of S_a. As in the definition of the ∂_w, this is independent of choice of reduced expression for w, up to sign. The degree of $\mathfrak{s}_w$ is $2\ell(w)$. Equation (3.2.2) implies

$$\partial_u \mathfrak{s}_w = \begin{cases} \pm \mathfrak{s}_{wu^{-1}} & \text{if } \ell(wu^{-1}) = \ell(w) - \ell(u), \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.7}$$

Lemma 3.2.1. Let $e \in S_a$ be the identity. Then $\mathfrak{s}_e = (-1)^{\binom{a}{3}}$.

Proof. This is a simple calculation, by induction on a. Alternatively, it follows from the graphical arguments in Proposition 3.5.6 (whose proof does not depend on the present claim). □

It follows that

$$\begin{aligned} &\text{for } \ell(w) < \ell(u), \quad (\underline{x}^A \partial_u)(\mathfrak{s}_w) = 0, \\ &\text{for } \ell(w) = \ell(u), \quad (\underline{x}^A \partial_u)(\mathfrak{s}_w) = \begin{cases} \pm \underline{x}^A & \text{if } w = u^{-1} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.2.8}$$

Proposition 3.2.2. There are no linear relations among the images of the $\{\underline{x}^A \partial_w\}_{w \in S_a, A \in \mathbb{Z}^n_{\geq 0}}$ in $\mathrm{End}(\mathrm{Pol}_a^{-1})$, nor among the images of the $\{\partial_w \underline{x}^A\}_{w \in S_a, A \in \mathbb{Z}^n_{\geq 0}}$. Thus the natural representations of the odd nilCoxeter and odd nilHecke rings on Pol_a^{-1} are faithful, and these rings have graded ranks

$$\begin{aligned} \mathrm{rk}_q(\mathrm{ONC}_a) &= q^{-\binom{a}{2}}[a]!, \\ \mathrm{rk}_q(\mathrm{ONH}_a) &= q^{-\binom{a}{2}}[a]!\frac{1}{(1-q^2)^a}. \end{aligned} \tag{3.2.9}$$

Proof. Since ONH_a is finite dimensional in each degree, the relations defining ONH_a imply that it suffices to prove the Proposition for either one of the two spanning sets; we do so for the set $\{\underline{x}^A\partial_w\}$. By equation (3.2.8),

$$(\underline{x}^A\partial_u)(\mathfrak{s}_e) = \begin{cases} (-1)^{\binom{a}{3}}\underline{x}^A & \text{if } u = e, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.10}$$

Thus no element $\underline{x}^A\partial_e$ is a linear combination of any other spanning set elements. Proceeding by induction on the Coxeter length of $w \in S_a$, suppose that for all $v \in S_a$ with $\ell(v) < \ell(w)$, no element $\underline{x}^A\partial_v$ is a linear combination of any other spanning set elements. Then by equation (3.2.8), if $\underline{x}^A\partial_u$ is a linear combination of other terms $\underline{x}^A\partial_v$, all the v must be of shorter Coxeter length than u; but by induction, there is no such relation. □

Our next goal is to express Pol_a^{-1} as a free left and right module over $O\Lambda_a$. The following lemma describes the basis we will use. Our proofs of Lemma 3.2.3 and Proposition 3.2.4 closely follow the proofs of Propositions 2.5.3 and 2.5.5 of [Man01], respectively.

Lemma 3.2.3. Let

$$\begin{aligned} \mathcal{H}_a &= \mathrm{span}_{\mathbb{Z}}\{\underline{x}^A \in \mathrm{Pol}_a^{-1} : A \leq \delta_a \text{ termwise}\} \\ &= \mathrm{span}_{\mathbb{Z}}\{x_1^{a_1}\cdots x_a^{a_a} : a_i \leq a-i \text{ for } 1 \leq i \leq a\}. \end{aligned} \tag{3.2.11}$$

Then the odd Schubert polynomials $\{\mathfrak{s}_w(x)\}_{w\in S_a}$ are an integral basis for $\mathcal{H}_a$.

Proof. It is immediate from the definition of the Schubert polynomials that they are all contained in $\mathcal{H}_a$, as the operators ∂_i only decrease exponents from $\underline{x}^{\delta_a}$. Since the set of Schubert polynomials and the defining basis of $\mathcal{H}_a$ both have $a!$ elements, it suffices to show linear independence and unimodularity. Suppose

$$\sum_{w\in S_a} c_w\mathfrak{s}_w(x) = 0 \tag{3.2.12}$$

with $c_w \in \mathbb{Z}$. Then by applying various operators ∂_u (as in the proof of Proposition 3.2.2), we see that all the $c_w = 0$, proving linear independence over $\mathbb{Q}$. So any $f \in \mathcal{H}_a$ is a rational linear combination of Schubert polynomials; but applying ∂_u's to an expression $f = \sum_w c_w \mathfrak{s}_w$, we see that each $c_w \in \mathbb{Z}$. $\square$

Proposition 3.2.4. Pol_a^{-1} is a free left and right $O\Lambda_a$-module of graded rank $q^{\binom{a}{2}}[a]!$, with a homogeneous basis given by the odd Schubert polynomials $\{\mathfrak{s}_w\}_{w \in S_a}$.

Proof. We will show that multiplication

$$O\Lambda_a \otimes \mathcal{H}_a \to \mathrm{Pol}_a^{-1} \tag{3.2.13}$$

is an isomorphism of abelian groups, realizing Pol_a^{-1} as a free left $O\Lambda_a$-module; a similar proof shows it is a free right module with the same basis.

Our first claim is that any $f \in \mathrm{Pol}_a^{-1}$ can be expressed in the form

$$f = \sum_{k=1}^{a-1} \sum_j \ell_{k,j} h_{k,j} x_a^k \qquad h_{k,j} \in \mathcal{H}_{a-1}, \quad \ell_{k,j} \in O\Lambda_a. \tag{3.2.14}$$

This being clear for $a = 1$, we proceed by induction. Expand a given $f \in \mathrm{Pol}_a^{-1}$ in powers of x_a,

$$f = \sum_k x_a^k f_k = \sum_k \sum_{i=1}^{a-2} \sum_j x_a^k \ell_{i,j,k} h_{i,j,k} x_{a-1}^i, \tag{3.2.15}$$

where $h_{i,j,k} \in \mathcal{H}_{a-2}$, $\ell_{i,j,k} \in O\Lambda_{a-1}$, and $f_k \in \mathrm{Pol}_{a-1}^{-1}$ for all i,j,k. Since $h_{i,j,k} x_{a-1}^i \in \mathcal{H}_{a-1}$, we can re-label and re-index to obtain an expression

$$f = \sum_{j,k} x_a^k \ell_{k,j} h_{k,j} \qquad h_{k,j} \in \mathcal{H}_{a-1}, \quad \ell_{k,j} \in O\Lambda_{a-1}. \tag{3.2.16}$$

By Corollary 3.1.8, each $x_a^k \ell_{k,j}$ can be re-written as a sum over terms of the form $x_a^{k'} \ell$, where $\ell \in O\Lambda_a$ and $1 \le k' \le a-1$. Re-indexing and collecting terms again, this proves the claim.

The above claim implies surjectivity of the multiplication map. Injectivity follows from a graded rank count. We have shown that Pol_a^{-1} is a free (left and right) $O\Lambda_a$-module of graded rank

$$(\mathrm{rk}_q)_{O\Lambda_a}(\mathrm{Pol}_a^{-1}) = q^{\binom{a}{2}}[a]!, \tag{3.2.17}$$

since the right-hand side is equal to $\sum_w q^{2\ell(w)} = \sum_w q^{\deg(\mathfrak{s}_w(x))}$. $\square$

We briefly recall the grading on matrix algebras over graded rings. Let $f(q)$ be a Laurent series in q and let $h(q)$ be a Laurent polynomial in q. If A is a graded ring with $\mathrm{rk}_q(A) = f(q)$ and M is a graded A-module with $(\mathrm{rk}_q)_A(M) = h(q)$, then

$$\mathrm{rk}_q(\mathrm{End}_A(M)) = f(q)h(q)h(q^{-1}). \tag{3.2.18}$$

Corollary 3.2.5. The natural action of ONH_a on Pol_a^{-1} by multiplication and odd divided difference operators is an isomorphism

$$\varphi:\ \mathrm{ONH}_a \xrightarrow{\cong} \mathrm{End}_{O\Lambda_a}(\mathrm{Pol}_a^{-1}) \cong \mathrm{Mat}_{q^{\binom{a}{2}}[a]!}(O\Lambda_a). \tag{3.2.19}$$

Proof. Since the ONH_a action is by linearly independent operators (see proof of Proposition 3.2.2), $\varphi \otimes_{\mathbb{Z}} \mathbb{Q}$ (and hence φ) is injective. Both ONH_a and $\mathrm{End}_{O\Lambda_a}(\mathrm{Pol}_a^{-1})$ have the same graded rank, so φ is surjective as well. $\square$

Proposition 3.2.6. For $a \geq 2$, the center of ONH_a is the ring of symmetric polynomials in the squared variables $x_1^2, \ldots, x_a^2$. This coincides with the center of $O\Lambda_a$.

Proof. Since ONH_a is a matrix ring over $O\Lambda_a$, their centers are the same when we view $O\Lambda_a \subset \mathrm{ONH}_a$ as scalar multiples of the identity. For the duration of this proof, then, we will just refer to "central" (skew) polynomials. First claim: *if f is central, then f is a polynomial in the squared variables $x_1^2, x_2^2, \ldots, x_a^2$.* To see this, fix some $1 \leq j \leq a$ and expand the skew polynomial f as a polynomial in x_j,

$$f = \sum_{k \geq 0} f_k x_j^k.$$

Then for each $k, \ell \geq 0$, let $f_{k,\ell}$ be the degree ℓ part of f_k, so

$$f = \sum_{k,\ell \geq 0} f_{k,\ell} x_j^k.$$

Multiplying this equation by x_j on the left and on the right and comparing the results, it follows that the degree of each $f_{k,\ell}$ must be even. Doing this for each j separately, the first claim follows.

Second claim: *if f is a polynomial in the squared variables, then $\partial_i(f) = 0$ if and only if $s_i(f) = f$.* First, suppose $s_i(f) = f$. Then f is a linear combination of terms which are a product of a factor not involving x_i, x_{i+1} and a factor of the form $x_i^{2k}x_{i+1}^{2\ell} + x_i^{2\ell}x_{i+1}^{2k}$. The operator ∂_i annihilates both sorts of factor, so $\partial_i(f) = 0$. Conversely, suppose $\partial_i(f) = 0$; without loss of generality suppose f is of homogeneous degree. Expand

$$f = \sum_{k,\ell \geq 0} f_{k,\ell} x_i^{2k} x_{i+1}^{2\ell},$$

where $f_{k,\ell}$ is a polynomial in the variables x_j for $j \neq i, i+1$. So

$$\begin{aligned} 0 = \partial_i(f) &= \sum_{k,\ell \geq 0} f_{k,\ell} \partial_i(x_i^{2k} x_{i+1}^{2\ell}) \\ &= \sum_{k,\ell \geq 0} (x_i^2 x_{i+1}^2)^\ell \left(f_{k,\ell} \partial_i(x_i^{2(k-\ell)}) + f_{\ell,k} \partial_i(x_{i+1}^{2(k-\ell)}) \right). \end{aligned}$$

By decreasing induction on $|k-\ell|$, this implies that $f_{k,\ell} = f_{\ell,k}$ for all k, ℓ, that is, $s_i(f) = f$.

We now use the second claim to show that a skew polynomial f is central if and only if it is a symmetric polynomial in the squared variables $x_1^2, \ldots, x_a^2$. By the second claim, f a symmetric polynomial in the squared variables if and only if it is a polynomial in the squared variables and $\partial_i(f) = 0$ for all i. For such an f,

$$(\partial_i f - f \partial_i)(g) = \partial_i(f) g + s_i(f) \partial_i(g) - f \partial_i(g) = 0,$$

so f commutes with all divided difference operators. So $f \in Z(\mathrm{ONH}_a)$ if f is a symmetric polynomial in the squared variables. Conversely, using the above observations, it is easy to see that all symmetric polynomials in the squared variables are in $Z(O\Lambda_a)$. $\square$

In $O\Lambda_a$ let $\widehat{e}_k$ be the k-th elementary polynomial in the squared variables x_j^2. Then $\Bbbk[\widehat{e}_1, \ldots, \widehat{e}_a]$ is a subalgebra isomorphic to Λ with gradings dilated by a factor of 2. By taking the limit defining $O\Lambda$, we have the following.

Corollary 3.2.7. The center of $O\Lambda$ is the polynomial subalgebra generated by $\{\widehat{e}_k\}_{k\geq 1}$.

3.2.1 Odd symmetrization

Lemma 3.2.8. The longest element of S_a acts on odd elementary polynomials as

$$(\varepsilon_k)^{w_0} = (-1)^{\binom{k}{2} + k\binom{a-1}{2}} \varepsilon_k. \tag{3.2.20}$$

In particular, $(O\Lambda_a)^{w_0} = O\Lambda_a$.

Proof. The set of monomials appearing in ε_k is unchanged by w_0, so we need only consider the incurred sign. The action on monomials is

$$x_{i_1} \cdots x_{i_k} \mapsto (-1)^{k\binom{a}{2}} x_{a+1-i_1} \cdots x_{a+1-i_k}.$$

The sign appears because elementary transpositions act on variables x_i with a minus sign. Next, a sign of $(-1)^{\binom{k}{2}}$ is incurred in sorting the right hand side into ascending order. Finally, remember that the monomials in ε_k appear with tildes, $\widetilde{x}_i = (-1)^{i-1} x_i$. The sign difference between removing the tildes on the left monomial and adding them in on the right monomial is $(-1)^{k(a-1)}$. Putting all these signs together, the sign is as described in the statement of the lemma. Since products of odd elementary polynomials are a basis of $O\Lambda_a$, it follows that $(O\Lambda_a)^{w_0} = O\Lambda_a$. $\square$

Remark 3.2.9. For $a > 2$ it is easy to see that given $s_j \in S_a$, the action by s_j does not preserve the ring of odd symmetric functions. For example

$$s_1(\varepsilon_1(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)) = \widetilde{x}_2 + \widetilde{x}_1 - \widetilde{x}_3,$$

which is not odd symmetric.

Recall our fixed choice of Coxeter word for the longest Weyl group element,

$$D_a = \partial_{w_0} = \partial_1(\partial_2\partial_1) \cdots (\partial_{a-1}\partial_{a-2} \cdots \partial_1). \tag{3.2.21}$$

A useful $\mathbb{Z}$-linear map is

$$\begin{aligned} \mathcal{S} &: \mathrm{Pol}_a^{-1} \to O\Lambda_a \\ f &\mapsto (-1)^{\binom{a}{3}} \left(D_a(f \underline{x}^{\delta_a}) \right)^{w_0}, \end{aligned} \tag{3.2.22}$$

which we call *odd symmetrization*. The name comes from the fact that, as we will prove in this section, $\mathcal{S}$ is the projection operator from Pol_a^{-1} onto its lowest-degree indecomposable summand, the subring $O\Lambda_a$. In order to prove this, we first establish a few lemmas.

A word $w = s_{i_1} \cdots s_{i_r}$ in the symmetric group S_a can act on a skew polynomial f in two ways:

- act by $\partial_{i_1} \cdots \partial_{i_r}$, as an element of the odd nilCoxeter ring,

- act by $s_{i_1} \cdots s_{i_r}$, as an element of S_a (equation (3.1.2)).

One way to hybridize and keep track of these two actions is to equip w with a function $\xi : \{1, 2, \ldots, r\} \to \{0, 1\}$ and to say

$$\text{the simple transposition } s_{i_j} \text{ acts as } \begin{cases} s_{i_j} & \text{if } \xi(j) = 0 \\ \partial_{i_j} & \text{if } \xi(j) = 1. \end{cases}$$

We will refer to the resulting operator as the *generalized action* of the pair (w, ξ), and denote its action on a polynomial f by $w^\xi \cdot f$. Give such a pair, define its *omission word* w^ξ_{om} to be the sub-word of w consisting of just those s_{i_j} such that $\xi(j) = 0$; that is, the sub-word of w corresponding to those transpositions which act via S_a rather than the odd nilCoxeter ring. In this language, the Leibniz rule (3.1.4) can be generalized,

$$\partial_w(fg) = \partial_{i_1} \cdots \partial_{i_r}(fg) = \sum_\xi (w^\xi \cdot f) \partial_{w^\xi_{\text{om}}}(g). \tag{3.2.23}$$

The sum is over all $2^{\ell(w)}$ possible choices of ξ. Note that the action of w^ξ is a generalized action, while the action of w^ξ_{om} is an action by odd divided difference operators.

Lemma 3.2.10 (Omission Word Lemma (OWL))**.** Suppose w is a reduced expression for w_0. For any ξ as above, either

1. $w^\xi \cdot f = 0$ for all $f \in O\Lambda_a$,
2. $\xi(j) = 0$ for all $j = 1, \ldots, r$, or
3. the omission word w^ξ_{om} is non-reduced.

In order to prove the OWL, we first introduce odd divided difference operators for non-adjacent transpositions. Notation: let $s_{k,\ell}$ be the transposition of k and ℓ in the symmetric group S_a, even if $|k - \ell| > 1$. For $1 \leq i, j \leq a$ and $i \neq j$, define the corresponding odd divided difference operator $\partial_{i,j}$ by

$$\partial_{i,j}(x_h) = \begin{cases} 1 & \text{if } h = i, j, \\ 0 & \text{if } h \neq i, j, \end{cases} \tag{3.2.24}$$

$$\partial_{i,j}(fg) = \partial_{i,j}(f)g + s_{i,j}(f)\partial_{i,j}(g).$$

Note that $\partial_{i,i+1} = \partial_i$ and that $\partial_{i,j}$ is homogeneous of degree -2. It is not true in general that the kernel of $\partial_{i,j}$ contains $O\Lambda_a$ (unless, of course, $j = i+1$).

Lemma 3.2.11. 1. For any $i \neq j$ and $k \neq \ell$, we have

$$\partial_{i,j} s_{k,\ell} + s_{k,\ell} \partial_{s_{k,\ell}(i,j)} = 0 \tag{3.2.25}$$

as operators on Pol_a^{-1}. By $s_{k,\ell}(i,j)$, we mean the pair obtained by applying the transposition $s_{k,\ell}$ to i and j. In particular, $\partial_{i,j}$ and $s_{k,\ell}$ anticommute if $\{i,j\} \cap \{k,\ell\} = \varnothing$.

2. If $\{i,j\} \cap \{k,\ell\} = \varnothing$, then

$$\partial_{i,j} \partial_{k,\ell} + \partial_{k,\ell} \partial_{i,j} = 0 \tag{3.2.26}$$

as operators on Pol_a^{-1}.

Proof. For both statements: first check on x_h and then induct using the Leibniz rule. □

Lemma 3.2.12. Suppose $1 \leq i < j \leq a$ and $f \in O\Lambda_a$. Then

$$\partial_{i+1} \partial_{i+2} \cdots \partial_{j-1} \partial_{i,j}(f) = 0. \tag{3.2.27}$$

Proof. The proof is by a slightly complicated induction. For now, consider all i, j simultaneously.

Step 1: We first prove the lemma in the case $f = \varepsilon_k$, $k \geq 0$. Let $D'_{i,j} = \partial_{i+1}\partial_{i+2} \cdots \partial_{j-1}\partial_{i,j}$. A divided difference operator $\partial_{p,q}$ applied to a monomial $x_J = x_{j_1} \cdots x_{j_\ell}$ vanishes if and only if x_p and x_q occur the same number of times in J. It follows that if the tuple J has no repeated entries, then $D'_{i,j}(x_J) \neq 0$ if and only if J contains either: all of $i+1, i+2, \ldots, j$ but not i, or all of $i, i+1, \ldots, j-1$ but not j. The terms of ε_k on which $D'_{i,j}$ does not vanish match into pairs

$$\widetilde{x}_I \widetilde{x}_{i+1} \cdots \widetilde{x}_j \widetilde{x}_J, \quad \widetilde{x}_I \widetilde{x}_i \cdots \widetilde{x}_{j-1} \widetilde{x}_J,$$

where I and J are tuples whose degrees sum to $k-(j-i)$. We compute

$$\begin{aligned}
&D'_{i,j}\left(\widetilde{x}_I\widetilde{x}_{i+1}\cdots\widetilde{x}_j\widetilde{x}_J+\widetilde{x}_I\widetilde{x}_i\cdots\widetilde{x}_{j-1}\widetilde{x}_J\right)\\
&\quad=\partial_{i+1}\cdots\partial_{j-1}\partial_{i,j}\left(\widetilde{x}_I\widetilde{x}_{i+1}\cdots\widetilde{x}_j\widetilde{x}_J+\widetilde{x}_I\widetilde{x}_i\cdots\widetilde{x}_{j-1}\widetilde{x}_J\right)\\
&\quad=\pm\partial_{i+1}\cdots\partial_{j-1}\partial_{i,j}\left(x_Ix_{i+1}\cdots x_jx_J+(-1)^\ell x_Ix_i\cdots x_{j-1}x_J\right)\\
&\quad=\pm\partial_{i+1}\cdots\partial_{j-1}\left((-1)^{|I|+\ell-1}x_Ix_{i+1}\cdots x_{j-1}x_J+(-1)^\ell(-1)^{|I|}x_Ix_{i+1}\cdots x_{j-1}x_J\right)\\
&\quad=0.
\end{aligned}$$

It follows that $D'_{i,j}(\varepsilon_k)=0$.

Step 2: We now induct on the degree of f. In each degree, we may assume f is a product $\varepsilon_{k_1}\cdots\varepsilon_{k_r}$. Step 1 covered the base cases $r=1$ and degree 1. In suffices, then, to take $f=gh$, where both g and h are odd symmetric and have positive degree. We will prove equation (3.2.27) simultaneously with the following claim: for $1\leq\ell\leq j-i$,

$$\partial_{i+1}\cdots\partial_{j-\ell}\left(s_{j-\ell+1}\cdots s_{j-1}s_{i,j}(g)\cdot\partial_{j-\ell+1}\cdots\partial_{j-1}\partial_{i,j}(h)\right)=0. \tag{3.2.28}$$

The $\ell=j-i$ case of (3.2.28),

$$s_{j-i+1}\cdots s_{j-1}s_{i,j}(g)\cdot\partial_{i+1}\cdots\partial_{j-1}\partial_{i,j}(h)=0,$$

follows from (3.2.27) in lower degree. Before proceeding, we fix i and induct on j, the base case $j=i+1$ being obvious.

Step 3: To prove (3.2.28) by decreasing induction on ℓ (keeping i,j both fixed), we compute

$$\begin{aligned}
&\partial_{i+1}\cdots\partial_{j-\ell}\left(s_{j-\ell+1}\cdots s_{j-1}s_{i,j}(g)\cdot\partial_{j-\ell+1}\cdots\partial_{j-1}\partial_{i,j}(h)\right)\\
&\qquad=\partial_{i+1}\cdots\partial_{j-\ell-1}(\pm s_{j-\ell+1}\cdots s_{j-1}s_{i,j}\partial_{i,j-\ell}(g)\cdot\partial_{j-\ell+1}\cdots\partial_{j-1}\partial_{i,j}(h)\\
&\qquad\quad+s_{j-\ell}\cdots s_{j-1}s_{i,j}(g)\cdot\partial_{j-\ell}\cdots\partial_{j-1}\partial_{i,j}(h)),
\end{aligned}$$

by the Leibniz rule and part 1 of Lemma 3.2.11. Any one of the operators $\partial_{i+1},\ldots\partial_{j-\ell-1}$ annihilates the second factor of the first term on the right-hand side by part 2 of Lemma 3.2.11, so this equals

$$\begin{aligned}
\ldots&=\pm s_{j-\ell+1}\cdots s_{j-1}s_{i,j}\partial_{i+1}\cdots\partial_{j-\ell-1}\partial_{i,j-\ell}(g)\cdot\partial_{j-\ell+1}\cdots\partial_{j-1}\partial_{i,j}(h)\\
&\quad+\partial_{i+1}\cdots\partial_{j-\ell-1}\left(s_{j-\ell}\cdots s_{j-1}s_{i,j}(g)\cdot\partial_{j-\ell}\cdots\partial_{j-1}\partial_{i,j}(h)\right).
\end{aligned}$$

This vanishes by induction: the first factor of the first term by (3.2.27) for lower j and the second term by (3.2.28) for higher ℓ. This proves (3.2.28) in this degree and for this ℓ, hence for all j, ℓ.

Step 4: It remains to prove (3.2.27). We have

$$\partial_{i+1} \cdots \partial_{j-1} \partial_{i,j}(gh) = \partial_{i+1} \cdots \partial_{j-1} \left(\partial_{i,j}(g) \cdot h\right) + \partial_{i+1} \cdots \partial_{j-1} \left(s_{i,j}(g) \cdot \partial_{i,j}(h)\right).$$

The second term on the right-hand side is zero by (3.2.28) at $\ell = 1$. Applying each of $\partial_{i+1}, \ldots \partial_{j-1}$ to the first term on the right-hand side and using the Leibniz rule, we always get zero for the term in which the divided difference operator hits h. Therefore this term equals $\partial_{i+1} \cdots \partial_{j-1} \partial_{i,j}(g) \cdot h$, which is zero by (3.2.27) in lower degree. This completes the proof. □

Proof of the OWL, Lemma 3.2.10. We are free to reorder D_a up to sign; we choose the ordering D'_a such that $D'_2 = \partial_1$ and $D'_a = \partial_{a-1} \cdots \partial_1 D'_{a-1}$. (In fact, $D'_a = (-1)^{\binom{a}{3}} D_a$.) Explicitly,

$$D_a = (\partial_{a-1} \cdots \partial_1) \cdots (\partial_{a-1}\partial_{a-2})\partial_{a-1}.$$

Fix some ξ and consider the generalized action $w^\xi \cdot f$. Let $s_{w_0,b}$ denote the longest element of the symmetric group among strands $a-b+1, \ldots, a$. If the rightmost ∂_{a-1} in D'_a acts with $\xi = 1$, then any f is annihilated and we are in Case 1. By induction on $b \geq 1$, then, suppose the bottom $\binom{b}{2}$ crossings all act with $\xi = 0$. The next b crossings are between strand pairs $(a-b, a-b+1), (a-b+1, a-b+2), \ldots, (a-1, a)$. Let i be the number of the left strand of any of these crossings which acts with $\xi = 1$. If $i < a-1$ and the $(i+1, i+2)$ crossing acts with $\xi = 0$, then the omission word w^ξ_{om} is non-reduced (pull a crossing s_{i+1} to the top of $s_{w_0,b}$), so we are in Case 3 and are done. We are therefore reduced to the case in which once one of these crossings acts with $\xi = 1$, then so do all others to the left. That is,

$$w^\xi \cdot f = (\ldots)\partial_{a-1} \cdots \partial_{a-b+\ell} s_{a-b+\ell-1} \cdots s_{a-b} s_{w_0,b}(f).$$

By part 1 of Lemma 3.2.11 this equals

$$\pm(\ldots) s_{a-b+\ell-1} \cdots s_{a-b} s_{w_0,b} \partial_{a-b+1} \cdots \partial_{a-b+\ell-1} \partial_{a-b,a-b+\ell}(f),$$

which vanishes by Lemma 3.2.12. □

An immediate corollary of the OWL is that for any $f \in O\Lambda_a$ and $g \in \mathrm{Pol}_a^{-1}$,

$$D_a(fg) = f^{w_0} D_a(g), \tag{3.2.29}$$

since $\partial_v = 0$ for any nonreduced word v.

Corollary 3.2.13. The odd symmetrization operator $\mathcal{S}$ is left $O\Lambda_a$-linear and is a projection operator onto the subring $O\Lambda_a \subset \mathrm{Pol}_a^{-1}$.

Proof. Left $O\Lambda_a$-linearity of $\mathcal{S}$ follows immediately from equation (3.2.29). The image of $\mathcal{S}$ is inside $O\Lambda_a$ due to the presence of the D_a in its definition and by Lemma 3.2.8. This image is all of $O\Lambda_a$ and $\mathcal{S}^2 = \mathcal{S}$ because, by the OWL and the calculation $D_a(\underline{x}^{\delta_a}) = (-1)^{\binom{a}{3}}$ (see (3.6.9)),

$$\mathcal{S}(f\underline{x}^{\delta_a}) = (-1)^{\binom{a}{3}} D_a(f\underline{x}^{\delta_a})^{w_0} = (-1)^{\binom{a}{3}} \left(f^{w_0} D_a(\underline{x}^{\delta_a})\right)^{w_0} = f \tag{3.2.30}$$

for any $f \in O\Lambda_a$. □

We conclude this section with another useful corollary of the OWL.

Corollary 3.2.14. For any $f \in \mathrm{Pol}_a^{-1}$,

$$D_a(f)^{w_0} = (-1)^{\binom{a}{2}} D_a(f^{w_0}). \tag{3.2.31}$$

Proof. Expand f in the odd Schubert polynomial basis,

$$f = \sum_{w \in S_a} f_w \mathfrak{s}_w,$$

where each $f_w \in O\Lambda_a$. Since the action of w_0 preserves $O\Lambda_a \subset \mathrm{Pol}_a^{-1}$, equation (3.2.29) implies

$$D_a(f)^{w_0} = \sum_w f_w D_a(\mathfrak{s}_w)^{w_0} = f_{w_0} D_a(\underline{x}^{\delta_a})^{w_0},$$

$$D_a(f^{w_0}) = \sum_w D_a((f_w)^{w_0}(\mathfrak{s}_w)^{w_0}) = f_{w_0} D_a((\underline{x}^{\delta_a})^{w_0}).$$

The last equality on each line is by degree reasons: the only Schubert polynomial of high enough degree not to be annihilated by D_a is $\mathfrak{s}_{w_0}$. The corollary then follows from the relations

$$D_a(\underline{x}^{\delta_a}) = (-1)^{\binom{a}{3}}, \quad D_a((\underline{x}^{\delta_a})^{w_0}) = (-1)^{\binom{a+1}{3}},$$

proved in Section 3.6. □

3.3 Odd Schur polynomials

By $P(a,b)$ we denote the set of all partitions α with at most a parts (that is, with $\alpha_{a+1}=0$) such that $\alpha_1 \leq b$. That is, $P(a,b)$ consists of partitions whose corresponding Young diagram fits into a box of size $a\times b$. Moreover, the set of all partitions with at most a parts (that is, the set $P(a,\infty)$) we denote simply by $P(a)$. $P(0)=\{\emptyset\}$ is the set of all partitions with at most 0 parts, so that $P(0)$ contains only the empty partition.

The cardinality of $P(a,b)$ is $\binom{a+b}{a}$. Its q-cardinality is a q-binomial coefficient,

$$|P(a,b)|_q := \sum_{\alpha\in P(a,b)} q^{2|\alpha|-ab} = \begin{bmatrix} a+b \\ a \end{bmatrix}. \tag{3.3.1}$$

The *dual* (or *conjugate*) partition of α, denoted $\overline{\alpha}=(\overline{\alpha}_1,\overline{\alpha}_2,\ldots)$ with $\overline{\alpha}_j=\#\{i|\alpha_i\geq j\}$, has Young diagram given by reflecting the Young diagram of α along the diagonal. For a partition $\alpha\in P(a,b)$ we define the *complementary partition* $\alpha^c=(b-\alpha_a,\ldots,b-\alpha_2,b-\alpha_1)$. To be more explicit we let $K=(b^a)\in P(a,b)$ and write $K-\alpha$ in place of α^c to emphasize that $\alpha\in P(a,b)$. Finally we define $\hat{\alpha}:=\overline{\alpha^c}$. Note that $\overline{\alpha}$ and $\hat{\alpha}$ belong to $P(b,a)$ and α^c to $P(a,b)$.

In the even case, one definition of the Schur polynomial corresponding to a partition α of length at most a is

$$s_\alpha(x_1,\ldots,x_a)=\partial_{w_0}(x^{\delta_a+\alpha})=\partial_{w_0}(x_1^{a-1+\alpha_1}x_2^{a-2+\alpha_2}\ldots x_a^{\alpha_a}). \tag{3.3.2}$$

In the odd case, we must be careful about the ordering of the terms in the above expression.

Definition 3.3.1. The *odd Schur polynomial* corresponding to a partition α of length at most a is the element of Pol_a^{-1} given by odd-symmetrizing the monomial $\underline{x}^\alpha$,

$$s_\alpha(x_1,\ldots,x_a)=\mathcal{S}(\underline{x}^\alpha)=(-1)^{\binom{a}{3}}D_a(\underline{x}^\alpha\underline{x}^{\delta_a})^{w_0}, \tag{3.3.3}$$

see also equation (3.2.22). As usual, $\underline{x}^\alpha=x_1^{\alpha_1}\cdots x_a^{\alpha_a}$.

Since $\mathcal{S}$ is the projection onto the subring of odd symmetric polynomials, s_α is odd symmetric.

Lemma 3.3.2. For $1\leq k\leq a$, we have

$$s_{(1^k)}(x_1,\ldots,x_a)=(-1)^{\binom{k}{2}}\varepsilon_k(x_1,\ldots,x_a). \tag{3.3.4}$$

Proof. Let $i_1 < \cdots < i_k$ be a subset of $\{1, \ldots, a\}$. Unless $i_1 = 1, \ldots, i_k = k$, the normal ordering of the monomial $\widetilde{x}_{i_1} \cdots \widetilde{x}_{i_k} \underline{x}^{\delta_a}$ will have a factor of the form $x_j^\ell x_{j+1}^\ell$ somewhere, so that $D_a(\widetilde{x}_{i_1} \cdots \widetilde{x}_{i_k} \underline{x}^{\delta_a}) = \partial_j(\widetilde{x}_{i_1} \cdots \widetilde{x}_{i_k} \underline{x}^{\delta_a}) = 0$. Hence, using odd-symmetrization and the definitions of ε_k and of $s_{(1^k)}$,

$$\varepsilon_k = \mathcal{S}(\varepsilon_k) = \mathcal{S}(\widetilde{x}_{i_1} \cdots \widetilde{x}_{i_k}) = (-1)^{\binom{k}{2}} s_{(1^k)}.$$

□

For a partition α, let $\frac{\alpha}{m}$ denote the partition whose Young diagram is obtained from that of α by removing the m-th and all lower rows; let $\frac{m}{\alpha}$ be likewise, removing the first through m-th rows.

Proposition 3.3.3 (Odd Pieri rule)**.** Let α be a partition and let $1 \leq k \leq a$. Then in $O\Lambda_a$,

$$s_\alpha s_{(1^k)} = \sum_\mu (-1)^{\left|\frac{i_1}{\alpha}\right| + \ldots + \left|\frac{i_k}{\alpha}\right|} s_\mu, \tag{3.3.5}$$

where the sum is over all partitions μ with Young diagram obtained from that of α by adding one box each to the rows $i_1 < \cdots < i_k$.

Note that mod 2, this is the usual Pieri rule. The diagrams μ described in the statement of the Proposition are precisely the diagrams obtained from α by what is called "adding a vertical strip of length k"; but we also need to know which rows we are adding to in order to get the correct sign.

Proof. Identifying $s_{(1^k)}$ with $(-1)^{\binom{k}{2}} \varepsilon_k$ by Lemma 3.3.2, we compute:

$$\begin{aligned} s_\alpha s_{(1^k)} &\overset{(3.3.3)}{=} (-1)^{\binom{a}{3}} D_a(\underline{x}^\alpha \underline{x}^{\delta_a})^{w_0} \cdot (-1)^{\binom{k}{2}} \varepsilon_k \\ &\overset{(3.2.20),(3.1.4)}{=} (-1)^{\binom{a}{3} + k\binom{a-1}{2}} D_a(\underline{x}^\alpha \underline{x}^{\delta_a} \varepsilon_k)^{w_0} \\ &\overset{(3.1.18)}{=} (-1)^{\binom{a}{3} + k\binom{a-1}{2}} \sum_{i_1 < \cdots < i_k} D_a(\underline{x}^\alpha \underline{x}^{\delta_a} \widetilde{x}_{i_1} \cdots \widetilde{x}_{i_k})^{w_0}. \end{aligned}$$

Commuting the factor $\widetilde{x}_{i_1} \cdots \widetilde{x}_{i_k}$ past $\underline{x}^{\delta_a}$ and normal ordering it with $\underline{x}^\alpha$ cancels the factor $(-1)^{k\binom{a-1}{2}}$ and introduces a factor of $(-1)^{\left|\frac{i_1}{\alpha}\right| + \ldots + \left|\frac{i_k}{\alpha}\right|}$ to each term. So we appear to have the desired equation (3.3.5), except that there are certain terms where μ is not non-decreasing, that is, does not correspond to a Young diagram (the term involving $x_{i_1} \cdots x_{i_k}$ corresponds to adding one box to α in each of rows $i_1, \ldots, i_k$). This occurs when μ is a composition but not a partition. A term will fail to correspond to a Young diagram if and only if the resulting monomial $\underline{x}^\alpha \underline{x}^{\delta_a} x_{i_1} \cdots x_{i_k}$, when normal ordered, has two adjacent exponents equal:

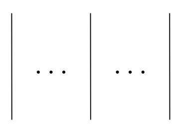
$$\underline{x}^\alpha \underline{x}^{\delta_a} x_{i_1} \cdots x_{i_k} = \pm \ldots x_j^m x_{j+1}^m \ldots \text{ for some } j, m \text{ if and only if } \mu \text{ is a composition but not a partition.}$$

Such a monomial is sent to zero by ∂_j. Re-ordering $D_a = \partial_{w_0}$ so as to have ∂_j act first, we see the term does not contribute to equation (3.3.5). ☐

3.4 Graphical calculus for the odd nilHecke algebra

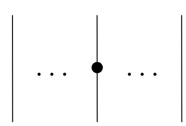
We find it convenient to use a graphical calculus to represent elements in ONH_a. The diagrammatic representation of elements in ONH_a is given by braid-like dotted diagrams D equipped with the height Morse function $h : D \to \mathbb{R}$, such that $h(g_1) \neq h(g_2)$ for any generators g_1, g_2 that appear in the diagram.

We write



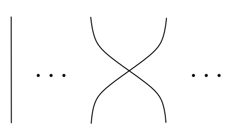
$$:= \quad 1 \in \mathrm{ONH}_a \tag{3.4.1}$$

with a total of a strands. The polynomial generators can be written as



$$:= \quad x_r \tag{3.4.2}$$

with the dot positioned on the r-th strand counting from the left, and

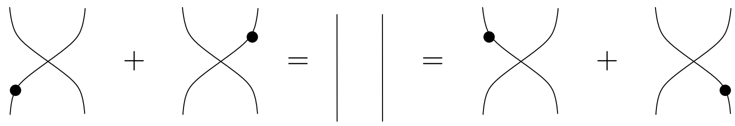


$$:= \quad \partial_r, \tag{3.4.3}$$

with the crossing interchanging the rth and $(r+1)$-st strands.

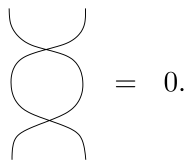
In the diagrammatic notation multiplication is given by stacking diagrams on top of each other, left-to-right becoming top-to-bottom. Relations in the odd nilHecke ring acquire a graphical interpretation. For example, the equalities $\partial_r x_r + x_{r+1}\partial_r = 1 = x_r\partial_r + \partial_r x_{r+1}$

become diagrammatic identities:



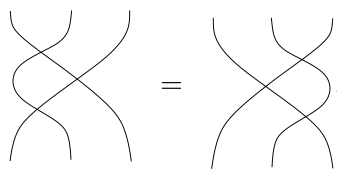
(3.4.4)

and the relation $\partial_r \partial_r = 0$ becomes



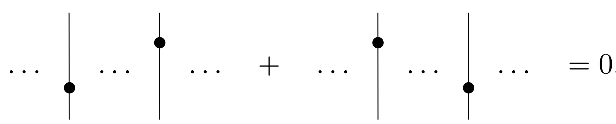
(3.4.5)

The relation $\partial_r \partial_{r+1} \partial_r = \partial_{r+1} \partial_r \partial_{r+1}$ is depicted as

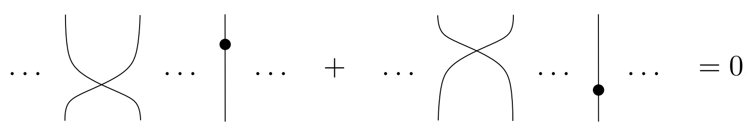


(3.4.6)

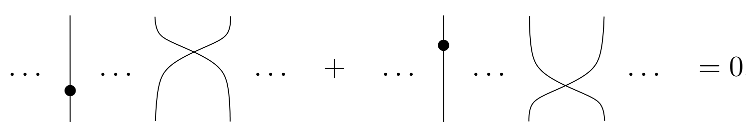
The remaining relations in the nilHecke ring can be encoded by the requirement that the diagrams super commute under braid-like isotopies:



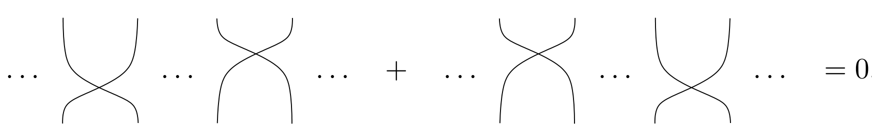
(3.4.7)



(3.4.8)



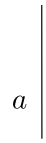
(3.4.9)



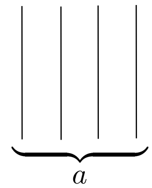
(3.4.10)

3.5 Box notation for odd thin calculus

To simplify diagrams, write



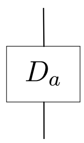
$$:=$$



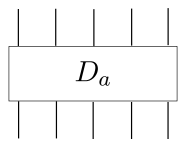
$$= \quad 1 \in \mathrm{ONH}_a,$$

where we will omit the label a if it appears in a coupon as below. The operator D_a is represented as

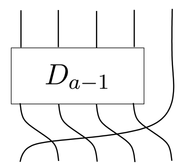
$$D_a \quad = $$



$$=$$



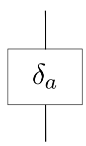
$$:=$$



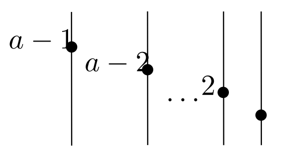
$$= \quad D_{a-1}(\partial_{a-1}\dots\partial_2\partial_1). \tag{3.5.1}$$

Next, let

$$\underline{x}^{\delta_a} \quad = $$



$$:=$$



$$= \quad x_1^{a-1}x_2^{a-2}\dots x_{a-2}^2 x_{a-1}.$$

Recall that the super degree of an element of ONH_a is the $\mathbb{Z}$-degree divided by 2. In particular, we have

$$\deg_s(D_a) \equiv \deg_s(\underline{x}^{\delta_a}) \equiv \binom{a}{2} \pmod 2. \tag{3.5.2}$$

Throughout this section we make frequent use of the trivial binomial identities

$$\binom{a}{2} = \sum_{j=1}^{a-1} j = \frac{a(a-1)}{2}, \tag{3.5.3}$$

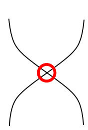
$$\binom{a}{3} = \sum_{j=1}^{a-1}\binom{j}{2} = \sum_{j=1}^{a-2}\sum_{\ell=1}^{j} j = \frac{a(a-1)(a-2)}{6}, \tag{3.5.4}$$

$$\binom{a+3}{4} = \sum_{k=1}^{a}\sum_{j=1}^{k}\sum_{\ell=1}^{j}\ell = \frac{a(a+1)(a+2)(a+3)}{24}. \tag{3.5.5}$$

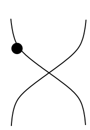
3.5.1 0-Hecke crossings

We will use a diagrammatic shorthand

$$\overline{\partial}_r \quad :=$$

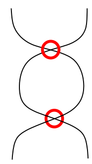
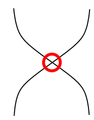


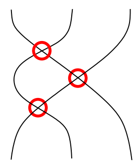
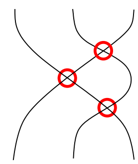
$$:=$$



$$= \quad x_r\partial_r. \tag{3.5.6}$$

It is easy to check that the elements $\overline{\partial}_r$ for $1 \leq r \leq a-1$ generate a copy of the 0-Hecke algebra:

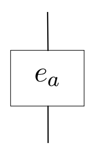
$\overline{\partial}_r^2$ =  =  = $\overline{\partial}_r$, (3.5.7)

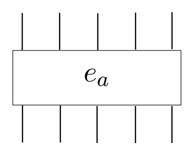
$\overline{\partial}_r\overline{\partial}_{r+1}\overline{\partial}_r$ =  =  = $\overline{\partial}_{r+1}\overline{\partial}_r\overline{\partial}_{r+1}$, (3.5.8)

and, since the element $\overline{\partial}_r$ has degree zero, we have

$$\overline{\partial}_r\overline{\partial}_s = \overline{\partial}_s\overline{\partial}_r \qquad \text{if } |r-s|>1.$$

Let w_0 denote a reduced word presentation of the longest element in S_a and define

e_a = 

= 

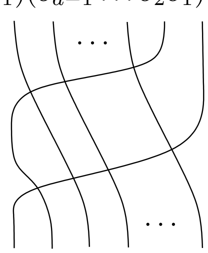
= $\overline{\partial}_{w_0}$.

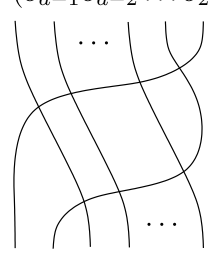
By the 0-Hecke relations above it is clear that $\overline{\partial}_{w_0}$ does not depend on the choice of reduced word presentation w_0.

3.5.2 Relations for D_a

Lemma 3.5.1 (Crossing slide lemma)**.**

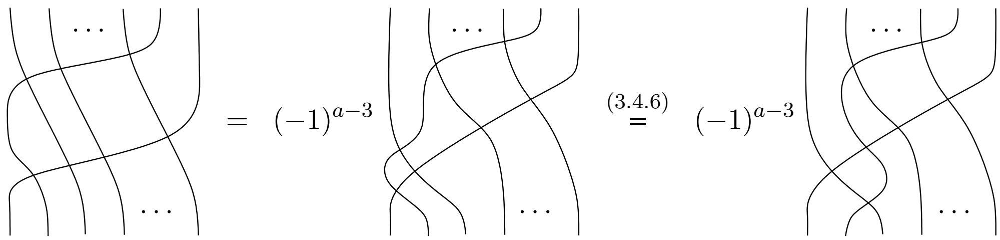
$$(\partial_{a-2}\dots\partial_2\partial_1)(\partial_{a-1}\dots\partial_2\partial_1) = (\partial_{a-1}\partial_{a-2}\dots\partial_2\partial_1)(\partial_{a-1}\dots\partial_2) \qquad (3.5.9)$$



=


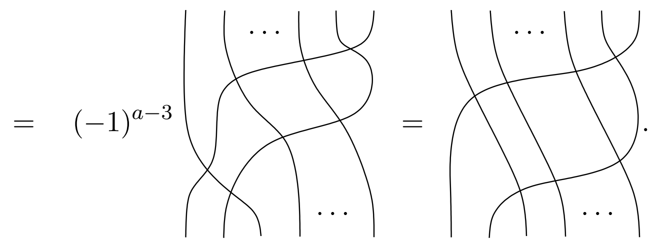
Proof. The proof is by induction on the number of strands. The base case of $a=3$ strands follows from the odd nilHecke relation (3.4.6). Assume the result holds for $a-1$ strands.

Then



(3.5.10)

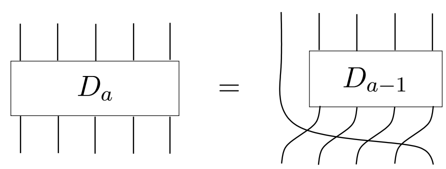
so that using the induction hypothesis the left side of (3.5.9) can be written as



(3.5.11)

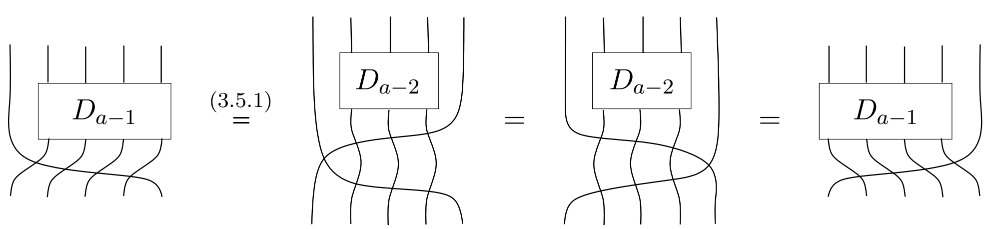
□

Lemma 3.5.2 (Alternative definition of D_a)**.** The element D_a defined in (3.5.1) can also be inductively defined by the equation



(3.5.12)

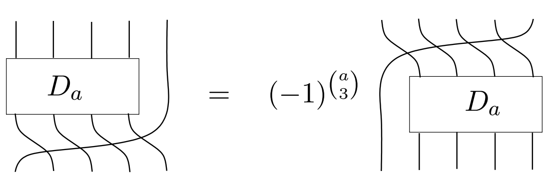
Proof. The proof is by induction with the base case following from (3.4.6). The left hand side can be re-written using the definition of D_a



(3.5.13)

where the second to last equality follows by repeatedly applying (3.4.6), and the last equality follows from the induction hypothesis. □

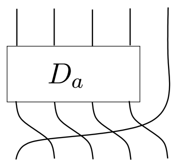
Lemma 3.5.3 (D_a slide)**.** For $a \geq 1$ the equation



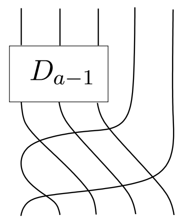
(3.5.14)

holds in ONH_a.

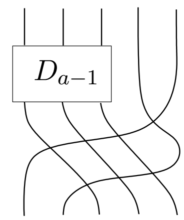
Proof. The proof is by induction on the number of strands. The base case is trivial, so we assume the result holds for up to $a-1$ strands.



$\overset{(3.5.1)}{=}$

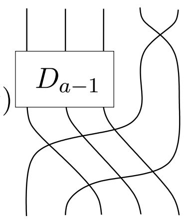


$\overset{(3.5.9)}{=}$

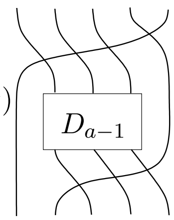


(3.5.15)

$= \quad (-1)^{\binom{a-1}{2}}$



$= \quad (-1)^{\binom{a-1}{2}+\binom{a-1}{3}}$



,(3.5.16)

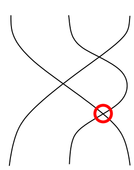
where we used the induction hypothesis in the last step. The result follows using the definition (3.5.1) of D_a and the identity

$$\binom{a-1}{2} + \binom{a-1}{3} = \binom{a}{3}. \tag{3.5.17}$$

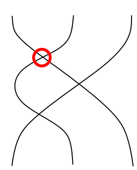
□

3.5.3 Relations involving the 0-Hecke generators

It is a simple calculation to see that

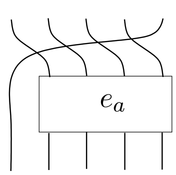


$=$

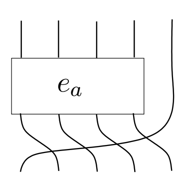


(3.5.18)

so that the relation



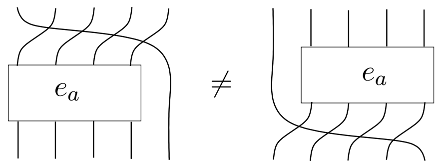
$=$



(3.5.19)

holds.

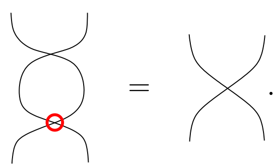
Remark 3.5.4. Note that an analogous relation



(3.5.20)

does not hold for sliding a projector e_a from the bottom left to the top right through a strand. This is because the reflection of equation (3.5.18) through a horizontal axis is false.

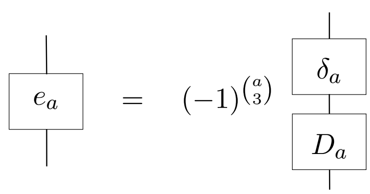
It follows from (3.4.4) that



(3.5.21)

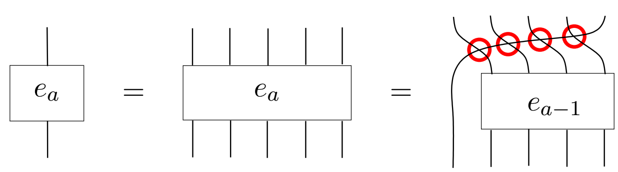
It will be convenient to express elements e_a in terms of one of the bases (3.2.4).

Proposition 3.5.5 (Projector in standard basis)**.**



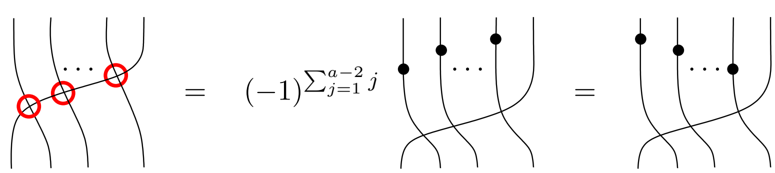
(3.5.22)

Proof. We prove the result by induction. The base case is trivial. Assume the result follows for e_{a-1}. From the definition of e_a we have



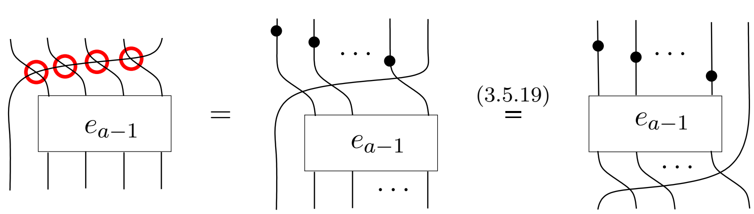
(3.5.23)

Sliding the dots from the 0-Hecke generators up to the top of the diagram we can write



(3.5.24)

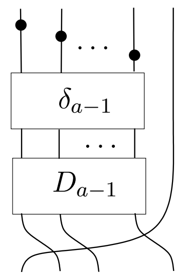
so that



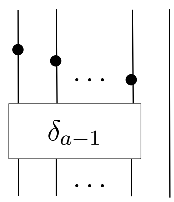
(3.5.25)

Using the induction hypothesis we write e_{a-1} in standard form

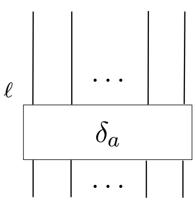
$$\dots \quad = \quad (-1)^{\binom{a-1}{3}} \qquad (3.5.26)$$



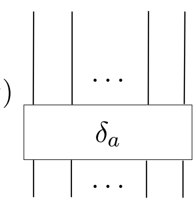
then slide the dots into the correct position using the identity



$$= \quad (-1)^{\sum_{j=1}^{a-2}\sum_{\ell=j}^{a-2}\ell}$$



$$= \quad (-1)^{\binom{a-1}{2}}$$



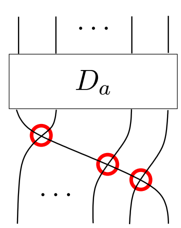
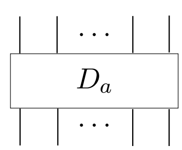
(3.5.27)

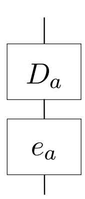
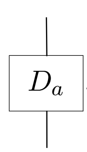
which follows by sliding the lower right most dot down past each of the $\sum_{j=1}^{a-2} j$ dots, then the second dot from the right, and so on. The Proposition follows from the inductive definition (3.5.1) of D_a and the binomial identity (3.5.17). ☐

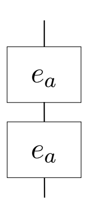
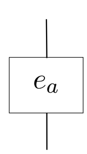
The following proposition shows that the elements e_a are idempotents in ONH_a.

Proposition 3.5.6.

The diagrammatic identities

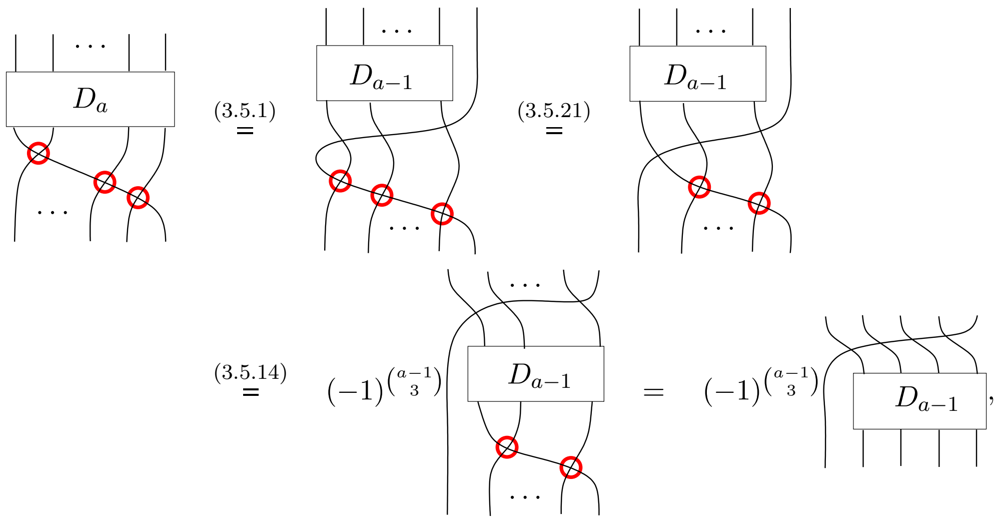
1.  $=$ ,

2.  $=$ ,

3.  $=$ 

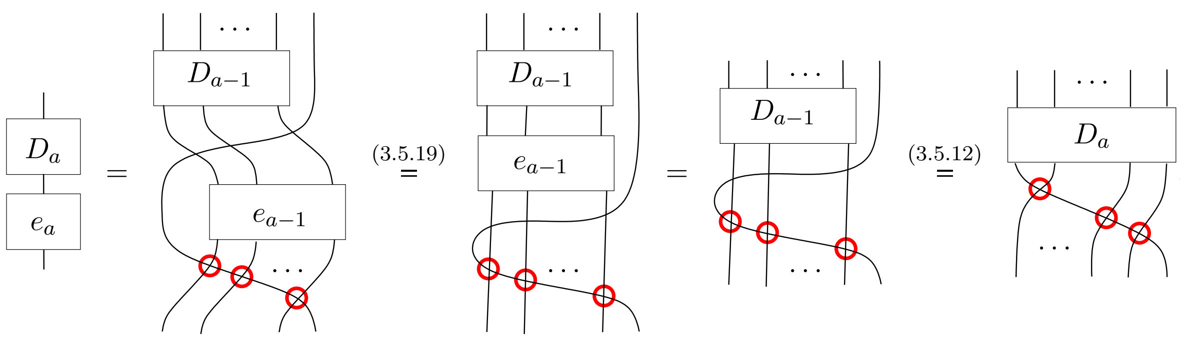
hold in ONH_a.

Proof. Part 1) follows from the computation



where the last equality follows by the induction hypothesis. This proves part 1) using equation (3.5.14) to slide D_{a-1} back through the line.

For the second claim observe that



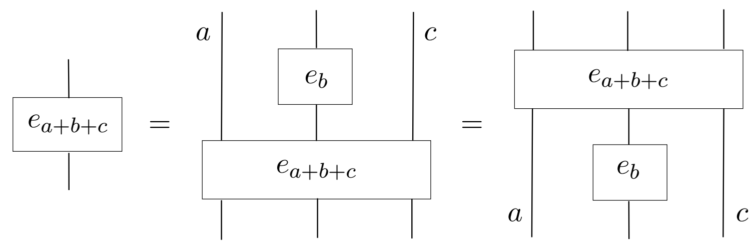
(3.5.28)

where the third equality follows by induction. The second claim follows from part 1). Part 3) follows from 2) using Proposition 3.5.5 since

$$e_a e_a = (-1)^{\binom{a}{3}} \underline{x}^{\delta_a} D_a e_a = (-1)^{\binom{a}{3}} \underline{x}^{\delta_a} D_a = e_a. \tag{3.5.29}$$

$\square$

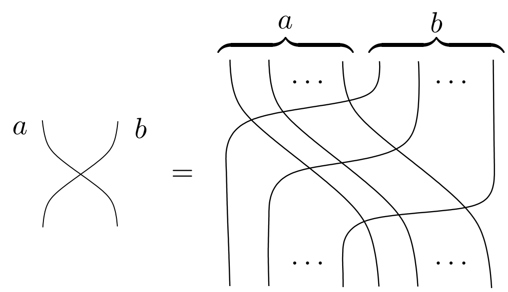
One can also prove the following equalities from the 0-Hecke relations.



(3.5.30)

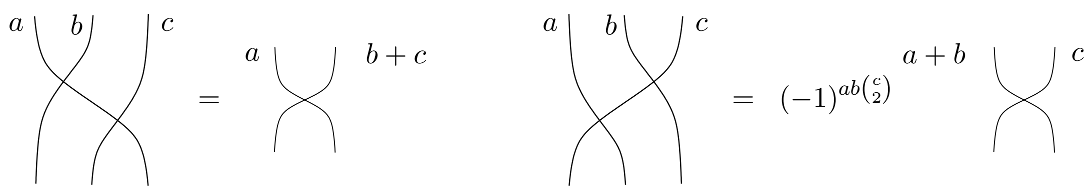
3.5.4 Crossings and projectors

Just as a vertical lines can be represented by a single line labelled a, we denote a specific choice of crossings of a strands with b strands as follows:



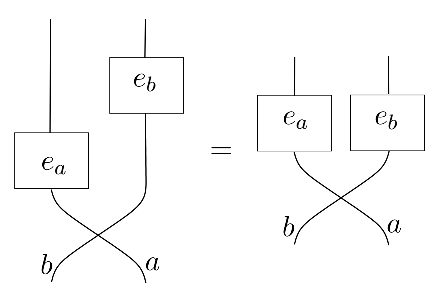
(3.5.31)

The first of the b strands crosses all of the a strands, then the second of the b strands crosses all of the a strands, and so on. One can easily verify that



(3.5.32)

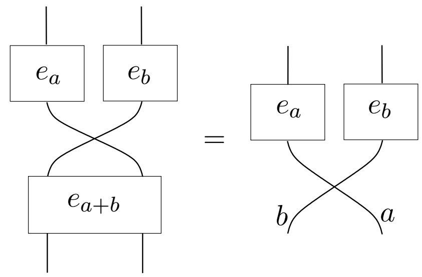
The respective heights of e_a and e_b in the diagram



(3.5.33)

are not relevant as these elements have even super-degrees and commute with each other.

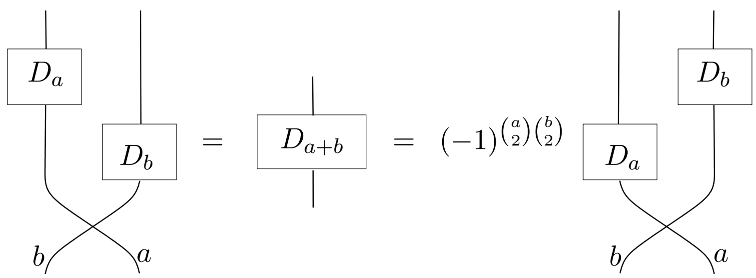
The equation



(3.5.34)

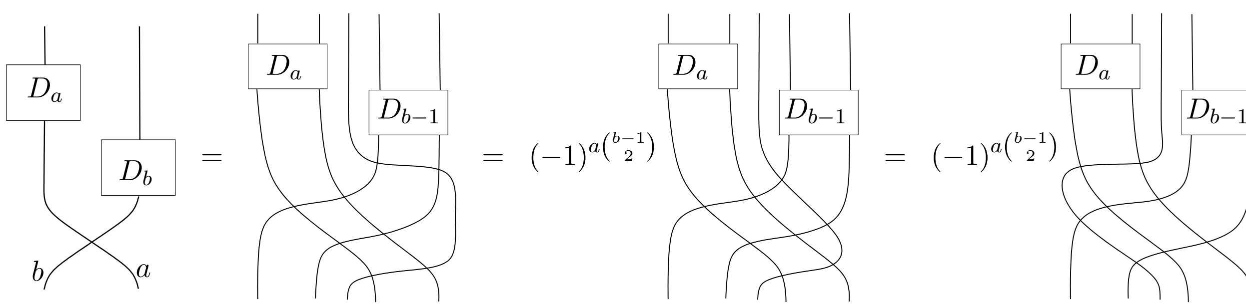
holds.

Proposition 3.5.7.

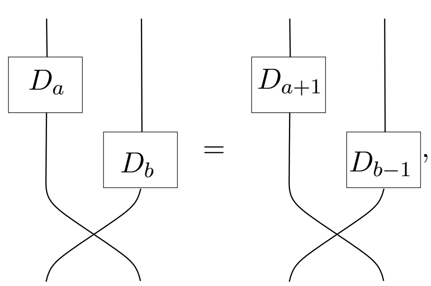


(3.5.35)

Proof. The second equality follows immediately from the first. To prove the first, observe that



where the sign in the second equality arises from sliding each of the $b-1$ crossings from D_b down. The third equality follows by repeated application of the Crossing Slide Lemma 3.5.1. Sliding the D_{b-1} down past the a crossings and using the inductive definition of D_{a+1} shows that



so that the result holds by induction. ∎

3.6 Automorphisms of the odd nilHecke algebra

Let $\sigma : \mathrm{ONH}_a \to \mathrm{ONH}_a$ denote the automorphism given by reflecting diagrams across the vertical axis. Let $\psi : \mathrm{ONH}_a \to \mathrm{ONH}_a^{\mathrm{op}}$ denote the anti-automorphism given by reflecting diagrams across the horizontal axis. These automorphisms commute with each other.

Below we collect the effect of applying these automorphisms to the elements $\underline{x}^{\delta_a}$, D_a, and e_a. One can check that

$$\begin{aligned} \underline{x}^{\delta_a} &= x_1^{a-1}x_2^{a-2}\dots x_{a-1}^1 x_a^0 && (3.6.1)\\ \sigma(\underline{x}^{\delta_a}) &= x_a^{a-1}x_{a-1}^{a-2}\dots x_2^1 x_1^0 && (3.6.2)\\ \psi(\underline{x}^{\delta_a}) &= x_a^0 x_{a-1}^1\dots x_2^{a-2} x_1^{a-1} && (3.6.3)\\ \psi\sigma(\underline{x}^{\delta_a}) &= x_1^0 x_2^1\dots x_{a-2}^{a-1} x_a^{a-1}. && (3.6.4) \end{aligned}$$

These monomials are related by the equations

$$\underline{x}^{\delta_a} = (-1)^{\binom{a}{4}}\psi(\underline{x}^{\delta_a}), \qquad \sigma(\underline{x}^{\delta_a}) = (-1)^{\binom{a}{4}}\sigma\psi(\underline{x}^{\delta_a}). \tag{3.6.5}$$

Similarly,

$$\begin{aligned} D_a &:= \partial_1(\partial_2\partial_1)\dots(\partial_{a-2}\dots\partial_1)(\partial_{a-1}\dots\partial_1)\\ \sigma(D_a) &= \partial_{a-1}(\partial_{a-2}\partial_{a-1})(\partial_2\dots\partial_{a-1})(\partial_1\dots\partial_{a-1})\\ \psi(D_a) &= (\partial_1\dots\partial_{a-1})(\partial_1\dots\partial_{a-2})\dots(\partial_1\partial_2)\partial_1\\ \psi\sigma(D_a) &= (\partial_{a-1}\dots\partial_1)(\partial_{a-1}\dots\partial_2)\dots(\partial_{a-1}\partial_{a-2})\partial_{a-1}, \end{aligned}$$

and it follows from Lemma 3.5.2 that

$$D_a = \sigma(D_a) = (-1)^{\binom{a-1}{4}}\psi(D_a) = (-1)^{\binom{a-1}{4}}\psi\sigma(D_a). \tag{3.6.6}$$

We have shown in Proposition 3.5.5 that

$$e_a = (-1)^{\binom{a+1}{4}+\binom{a}{4}}\underline{x}^{\delta_a}D_a = (-1)^{\binom{a}{3}}\underline{x}^{\delta_a}D_a \tag{3.6.7}$$

It is also worth while to write this equation in another form,

$$e_a = (-1)^{\binom{a+1}{4}}\psi(\underline{x}^{\delta_a})D_a. \tag{3.6.8}$$

It follows from Proposition 3.5.6 that

$$D_a(\underline{x}^{\delta_a}) = (-1)^{\binom{a}{3}} \qquad \text{and} \qquad D_a(\psi(\underline{x}^{\delta_a})) = (-1)^{\binom{a+1}{4}}. \tag{3.6.9}$$

3.7 Applications to odd Schur functions

The results of this section all come from [Ell12].

3.7.1 Equivalence of definitions of odd Schur functions

In Section 2.2.4 we defined odd Schur functions in $O\Lambda$ using odd Kostka numbers. Then, in Section 3.3, we defined Schur polynomials in the finite variable quotient $O\Lambda_a$ using odd symmetrization. It is natural to ask whether the former, in the finite variable quotient, coincides with the latter. In fact the two notions do coincide. To prove this, in this section we use signed plactic relations to introduce a third definition and show the other two are each equivalent to the plactic one.

3.7.1.1 Odd plactic Schur polynomials

Let $A = \{a_1, a_2, \ldots\}$ be an ordered alphabet. In practice, we will take $A = \mathbb{Z}_{>0}$ when working with $O\Lambda$ and $A = \{1, 2, \ldots, n\}$ when working with $O\Lambda_n$. In order to add, multiply, and assign signs to tableaux, we will use the *odd plactic ring* $\mathbb{Z}Pl$, which is the unital ring defined by

$$\begin{aligned} &\text{generators:} && A && && \\ &\text{relations:} && yzx = -yxz && \text{if } x < y \leq z && (K'), \\ & && xzy = -zxy && \text{if } x \leq y < z && (K''). \end{aligned} \tag{3.7.1}$$

When we want to emphasize that the alphabet in question is $\{1, 2, \ldots, n\}$, we will sometimes write $\mathbb{Z}Pl_n$ instead of $\mathbb{Z}Pl$. The relations $(K'), (K'')$ are called elementary Knuth transformations. We define a map from the set of semistandard Young tableaux with entries in the alphabet A to the odd plactic ring by

$$\begin{aligned} \{\text{SSYTs}\} &\to \mathbb{Z}Pl, \\ T &\mapsto w_r(T). \end{aligned} \tag{3.7.2}$$

Since both the relations $(K'), (K'')$ are transpositions of letters with a minus sign, $\mathbb{Z}Pl_n$ sits as an intermediate quotient between a free algebra and the skew polynomial ring,

$$\mathbb{Z}\langle x_1, \ldots, x_n \rangle \twoheadrightarrow \mathbb{Z}Pl_n \twoheadrightarrow \mathrm{Pol}_n^{-1},$$

where the first map sends $\widetilde{x}_i$ to i and the second map sends i to $\widetilde{x}_i$. If w is a word in $\mathbb{Z}Pl_n$, we will write $\widetilde{x}^w$ for the image of w in Pol_n^{-1}. In particular, a semistandard Young tableau T is sent to $\widetilde{x}^{w_r(T)}$.

The utility of the plactic ring is in large part due to the following remarkable theorem.

Theorem 3.7.1 ([Ful97], Section 2.1)**.** Up to sign, every word is equivalent via relations (K') and (K'') to the row word $w_r(T)$ of a unique tableau T.

Thus the set of all Young tableaux with entries in A forms a basis of $\mathbb{Z}Pl$. We will informally refer to the multiplication of tableaux in the following; what we mean is the multiplication of their row words in $\mathbb{Z}Pl$. In terms of tableaux, the relations (K') and (K'') can be interpreted as "bumping transformations":

$$(K') \qquad \boxed{y\,|\,z}\cdot\boxed{x} = -\begin{array}{|c|c|}\hline x & z \\ \hline y & \\ \cline{1-1}\end{array} \qquad \text{if } x < y \leq z,$$

$$(K'') \qquad \boxed{x\,|\,z}\cdot\boxed{y} = -\begin{array}{|c|c|}\hline x & y \\ \hline z & \\ \cline{1-1}\end{array} \qquad \text{if } x \leq y < z.$$

For a detailed exposition of bumping, see Section 1.1 of [Ful97].

If a word w is known to be the row word of some tableau, then it is easy to reconstruct the tableau from the word. Since the row entries of a tableau never decrease from left to right and the column entries must always increase from top to bottom, reading the word w from left to right until the first adjacent decreasing pair simply gives the bottom row of the tableau. Then continuing to read until the next adjacent decreasing pair gives the second to bottom row, and so forth.

Example 3.7.2. Using $\mathbb{Z}_{>0}$ as the ordered alphabet,

$$w = 53422331112 \qquad \text{corresponds to} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 3 \\ \hline 3 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 5 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}.$$

Definition 3.7.3. Let λ be a partition. Define an element of $\mathbb{Z}Pl_n$ by

$$\widehat{s}_\lambda = (-1)^{\binom{\lambda}{2}+N(\lambda)} \sum_{T\in SSYT(\lambda)} T. \tag{3.7.3}$$

Its image in Pol_n^{-1},

$$s_\lambda^p = (-1)^{\binom{\lambda}{2}+N(\lambda)} \sum_{T\in SSYT(\lambda)} \widetilde{x}^{w_r(T)}, \tag{3.7.4}$$

is called the *plactic odd Schur function* corresponding to λ.

The label p in the notation for the plactic odd Schur function is to distinguish it from the two definitions already introduced. Until Theorem 3.7.7, which will prove that all three are equivalent, we will write s_λ^s for the odd Schur polynomials defined by odd symmetrization (3.3.3) and s_λ^K for the odd Schur functions defined using odd Kostka numbers (2.2.16).

Remark 3.7.4. For the rest of the paper, we will not always specify whether we are working in $O\Lambda$ or in $O\Lambda_n$. Generally speaking, results will hold in $O\Lambda$. The only change required in passing to $O\Lambda_n$ is the understanding that certain elements become zero. It is easy to see that $O\Lambda_n = O\Lambda/(e_m : m > n)$, and it will follow from Theorem 3.7.7 that $s_\lambda = 0$ in $O\Lambda_n$ if and only if λ has height greater than n. With this understood, the proofs and results in the rest of this paper work in either context.

Lemma 3.7.5. If $\lambda = (1^k)$, then up to sign, all three Schur functions coincide with the corresponding elementary polynomial:

$$s^K_{(1^k)} = s^p_{(1^k)} = s^s_{(1^k)} = (-1)^{\binom{k}{2}} e_k. \tag{3.7.5}$$

If $\lambda = (k)$, the combinatorial and plactic Schur functions coincide with the corresponding complete polynomial:

$$s^K_{(k)} = s^p_{(k)} = h_k. \tag{3.7.6}$$

The last equation equals $s^s_{(k)}$ too, but we will prove this later.

Proof. The equality $s^K_{(1^k)} = (-1)^{\binom{k}{2}} e_k$ follows from Proposition 2.2.19 and the equality $s^s_{(1^k)} = (-1)^{\binom{k}{2}} e_k$ is Lemma 3.3.2. Since the row word of a semistandard Young tableau on the shape (1^k) is $i_k \cdots i_1$ for positive integers $i_1 < \ldots < i_k$ and it takes $\binom{k}{2}$ transpositions to ascend-sort the monomial $\widetilde{x}_{i_k} \cdots \widetilde{x}_{i_1}$, the equality with $s^p_{(1^k)}$ holds. It is similar but easier to show $s^p_{(k)} = h_k$, and $s^K_{(k)} = h_k$ is obvious because the only semistandard Young tableau of content (k) is a row of 1's. ∎

3.7.1.2 Comparison with previous definitions

For a partition λ, let $\frac{i}{\lambda}$ be the Young diagram obtained by removing rows 1 through i from the diagram corresponding to λ. Similarly, let $\frac{\lambda}{i}$, $i|\lambda$, and $\lambda|i$ be obtained by removing rows i through the bottom, columns 1 through i, and columns i through the rightmost respectively. We say that a skew shape is a *vertical strip* (respectively *horizontal strip*) if no two of its boxes are in the same row (respectively column). We say that a diagram μ is obtained from λ by adding a vertical strip if $\lambda \subset \mu$ and μ/λ is a vertical strip; likewise for horizontal strips.

The odd Pieri rule of Proposition 3.3.3 reads, in the notation of this section,

$$s_\lambda^s s_{(1^k)}^s = \sum_\mu (-1)^{\left|\frac{i_1}{\lambda}\right|+\ldots+\left|\frac{i_k}{\lambda}\right|} s_\mu^s. \tag{3.7.7}$$

We refer to this as the "e-right odd symmetrzied Pieri rule." The sum is over all μ obtained from λ by adding a vertical strip of size k, and $i_1, \ldots, i_k$ are the rows of λ to which a box was added.

The plactic odd Schur functions satisfy the same relation. We prove the horizontal strip variant instead, for simplicity.

Proposition 3.7.6 (Odd Pieri rule, h-right plactic version)**.** Let λ be a partition. Then

$$(-1)^{NE(\lambda)} s_\lambda^p s_{(k)}^p = \sum_\mu (-1)^{NE(\mu)} (-1)^{|i_1|\lambda|+\ldots+|i_k|\lambda|} s_\mu^p. \tag{3.7.8}$$

The sum is over all μ obtained from λ by adding a horizontal strip of size k, and $i_1, \ldots, i_k$ are the columns of λ to which a box was added.

Proof. Expanding all Schur functions as odd plactic sums, we want to prove

$$(-1)^{\binom{\lambda}{2}+N(\lambda)+NE(\lambda)} \widetilde{x}^{w_r(T)} \widetilde{x}^{w_r(V)} = (-1)^{\binom{\mu}{2}+N(\mu)+NE(\mu)} (-1)^{|i_1|\lambda|+\ldots+|i_k|\lambda|} \widetilde{x}^{w_r(U)}$$

whenever T is a Young tableau of shape λ, V is a Young tableau of shape (k), $TV = U$ in the even plactic ring with $\mathrm{sh}(U) = \mu$, and the boxes of μ/λ are in columns $i_1, \ldots, i_k$. This is because mod 2 the even and odd plactic rings are isomorphic (by the obvious map, $w_r(T) \mapsto w_r(T)$), so the set of products $\widetilde{x}^{w_r(T)} \widetilde{x}^{w_r(V)}$ and the set of terms $\widetilde{x}^{w_r(U)}$ which occur on the right-hand side of (3.7.8) are in bijection, with (T, V) corresponding to U if and only if $TV = U$ in the even plactic ring. Suppose the leftmost box of V has entry j. If that box ends up in column i, we claim

$$(-1)^{\binom{\lambda}{2}+N(\lambda)+NE(\lambda)} \widetilde{x}^{w_r(T)} \widetilde{x}_j = (-1)^{\binom{\mu}{2}+N(\mu)+NE(\mu)+|i|\lambda|} \widetilde{x}^{w_r(U)}.$$

For any partition ν, $(-1)^{\binom{\nu}{2}+N(\nu)+NE(\nu)} = (-1)^{NW(\nu)}$. The new box of μ is not northwest of any other box (since it must be a southeast corner), so the sign discrepancy only counts those boxes northwest of the new box. The sign $(-1)^{|i|\lambda|}$ counts boxes northeast of the new box, so the overall sign is $(-1)^{\sum_j(\lambda_j-1)}$. And this is precisely the sign between $\widetilde{x}^{w_r(T)}\widetilde{x}_j$ and $\widetilde{x}^{w_r(U)}$, since bumping a box past a row of length r incurs a sign of $(-1)^{r-1}$. Finally, note that as the boxes of V are added one at a time, the signs cancel telescopically so as to yield the sign of equation (3.7.8). $\square$

We now have the tools necessary to prove the main result of this section.

Theorem 3.7.7. The three notions of odd Schur function all coincide: for any partition λ,

$$s_\lambda^K = s_\lambda^p = s_\lambda^s. \tag{3.7.9}$$

Proof. The proof that $s_\lambda^K = s_\lambda^p$ is similar in spirit to the proof of the odd Pieri rule, h-right plactic version. By the same sort of analysis as in that proof, one shows that for a partition $\mu = (\mu_1, \ldots, \mu_r)$,

$$\begin{aligned} h_\mu = h_{\mu_1}\cdots h_{\mu_r} &= \sum_{\mathrm{ct}(T)=\mu} (-1)^{NE(T)+NE^<(T)+\binom{T}{2}+N(T)}\widetilde{x}^{w_r(T)} \\ &= \sum_\lambda \sum_{T\in SSYT(\lambda,\mu)} (-1)^{NE(\lambda)+NE^<(T)} s_\lambda^p. \end{aligned}$$

Since $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda^K$ and both $\{h_\mu\}, \{s_\lambda^K\}$ are integral bases of the ring of odd symmetric functions, it suffices to check

$$(-1)^{NE(\lambda)+NE^<(T)} = (-1)^{N(\lambda)+N^<(T)}$$

whenever T is a Young tableau of shape λ (the right-hand side is the sign with which T is counted in defining $K_{\lambda\mu}$). The sum computing the sign from a particular box B of T involves two types of other box: those northwest or directly north, and those northeast of B. For those northeast, the sign is identical. Those northwest or directly north are ignored by the left-hand sign, but the entry of such a box is necessarily less than that of B (since T is semistandard), so the right-hand sign is $+1$. Hence $s_\lambda^K = s_\lambda^p$.

We now use the Pieri rules (Propositions 3.3.3, 3.7.6) to prove $s_\lambda^p = s_\lambda^s$. For $\lambda = (1^k)$, this is true by Lemma 3.7.5. Using this as a base case, we now induct on the width of λ, and within each particular width we induct with respect to the lexicographic order of λ^T. Since we have shown $s_\lambda^K = s_\lambda^p$, we can apply the involution $\psi_1\psi_2$ to equation (3.7.8) to show the plactic Schur functions obey an e-right Pieri rule of the same form and signs as the one obeyed by the Schur functions s_λ^K,

$$s_\lambda^p s_{(1^k)}^p = \sum_\mu (-1)^{\left|\frac{i_1}{\lambda}\right|+\ldots+\left|\frac{i_k}{\lambda}\right|} s_\mu^p. \tag{3.7.10}$$

Let λ be a partition of width $\lambda_1 = r$. Using the e-right Pieri rule, both types of Schur function satisfy

$$s_{(\lambda_1^T)} \cdots s_{(\lambda_r^T)} = \sum_\mu \pm s_\mu,$$

where each μ has width at most r and is lexicographically greater than or equal to λ. The coefficient of s_λ on the right-hand side is ± 1. All the signs $\pm$ are the same for the two types of Schur function, so this allows us to solve for both s_λ^p, s_λ^s in terms of elementary functions by the same expressions; hence $s_\lambda^p = s_\lambda^s$. $\square$

Corollary 3.7.8. The span of the $\widehat{s}_\lambda$ in $\mathbb{Z}Pl_n$ is a subalgebra isomorphic to $O\Lambda_n$, and this subalgebra is taken isomorphically onto $O\Lambda_n \subset \mathrm{Pol}_n^{-1}$ by the map $w \mapsto \widetilde{x}^w$.

For the rest of this paper, we will drop the superscript labels on odd Schur functions.

3.7.2 The even Littlewood-Richardson rule

This subsection is a review of the classical Littlewood-Richardson rule. Nothing is new, and its only role is to fix notation and conventions.

For this subsection, we work in the even setting.

Let μ, ν, λ be partitions. The *Littlewood-Richardson coefficient* $c_{\mu\nu}^\lambda$ is the coefficient of s_λ in $s_\mu s_\nu$,

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda.$$

If $|\mu| + |\nu| \neq |\lambda|$, then $c^{\lambda}_{\mu\nu} = 0$. Schur functions are generating functions for semistandard Young tableaux of a given shape,

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x^{w_r(T)}.$$

It follows that, for any fixed semistandard Young tableau T_0 of shape λ,

$$c^{\lambda}_{\mu\nu} = \#\{U \in SSYT(\mu), V \in SSYT(\nu) : UV = T_0\}. \tag{3.7.11}$$

Here, the product of tableaux is taken in the (even) plactic ring. If $T_0 = T_\lambda$, then it is not hard to see that $UV = T_\lambda$ forces V to be T_ν; equation (3.7.11) becomes

$$c^{\lambda}_{\mu\nu} = \#\{U \in SSYT(\mu) : UT_\nu = T_\lambda\}. \tag{3.7.12}$$

Since computing the set on the right-hand side of (3.7.12) can be tricky in practice, we would like to have a simpler combinatorial description of the $c^{\lambda}_{\mu\nu}$. The Littlewood-Richardson rule provides one of many such simpler descriptions. Good accounts of the rule and its proof are given in [Ful97, Chapter 5] and [Sta99, Appendix A1]. We will review the terminology and statement here.

Recall that a *(Young) skew shape* λ/μ is the complement of a subdiagram $\mu \subseteq \lambda$. A *semistandard skew tableau* is a skew shape which has been filled with entries from some ordered alphabet, subject to the same rules as for a semistandard tableau: entries must strictly increase in columns (top to bottom) and must not decrease in rows (left to right). We write $SSYT(\lambda/\mu)$ for the set of semistandard skew tableaux of shape λ/μ and $SSYT(\lambda/\mu, \nu)$ for the set of semistandard skew tableaux of shape λ/μ and content ν.

A word $w = w_1 \cdots w_r$ in some ordered alphabet is called *Yamanouchi* (or a *reverse lattice word*) if, when read backwards, each initial subword has at least as many a's as b's whenever $a < b$. For example, 312211 is Yamanouchi but 1221 and 112 are not. A skew tableau S is called a *Littlewood-Richardson tableau* if $w_r(S)$ is a Yamanouchi word. The following is Proposition 3, Chapter 5 of [Ful97] and Theorem A1.3.3 of [Sta99].

Theorem 3.7.9 ((Even) Littlewood-Richardson Rule)**.** The coefficient $c^{\lambda}_{\mu\nu}$ equals the number of Littlewood-Richardson tableaux S of shape λ/μ and content ν.

One specific bijection between the set described in the theorem and the set in equation (3.7.12) is described in the following section, where we use it to deduce an odd analogue of Theorem 3.7.9.

Example 3.7.10. Let $\mu \subseteq \lambda$ be a subdiagram and let $k = |\lambda| - |\mu| \geq 1$. If S is a Littlewood-Richardson tableau of shape λ/μ and content (k), then no two boxes of S can be in the same column (since all entries equal 1). And on any such skew shape λ/μ, there is exactly one tableau of content (k). Thus $c^{\lambda}_{\mu(k)} = 1$ if λ/μ is a horizontal strip and equals 0 otherwise. We have deduced the (even) Pieri rule,

$$s_\mu s_{(k)} = \sum_{\substack{\lambda \\ \lambda/\mu \text{ is a} \\ \text{horizontal strip}}} s_\lambda. \tag{3.7.13}$$

Using the standard involution ω on Λ, or just arguing in analogy with the above, the same is true if (k) is replaced by (1^k) and "horizontal" is replaced by "vertical."

Example 3.7.11. The lowest degree product which is not described by the Pieri rule is

$$s_{21}s_{21} = s_{2211} + s_{222} + s_{3111} + 2s_{321} + s_{33} + s_{411} + s_{42}.$$

3.7.3 The odd Littlewood-Richardson rule

The sign of a Young tableau T is the sign between its row word monomial $\widetilde{x}^{w_r(T)}$ and the ascend-sorting of that monomial. As explained in Section 2.2.4, this sign equals $(-1)^{N^<(T)}$. If S is a skew tableau of shape λ/μ, let j be an element of the alphabet less than every entry of S and let $\widehat{S}$ be the Young tableau of shape λ formed by placing j in each box of μ and filling the rest so as to match S. We then define the sign of S to be

$$\text{sign}(S) = \text{sign}(\widehat{S}) = N^<(\widehat{S}).$$

When $\mu = (0)$, this reduces to the sign of a tableau as defined earlier. More generally, whenever we write a count $N(S), N^<(S), \ldots$, read $\widehat{S}$ for S.

Example 3.7.12. To either alphabet $\mathbb{Z}_{>0}$ or $\{1, 2, \ldots, n\}$, we can always adjoin 0 and take $j = 0$. If $\lambda = (3, 3, 2, 1)$, $\mu = (2, 1, 1)$, and

$$S = \begin{array}{ccc} & & \boxed{1} \\ & \boxed{1} & \boxed{2} \\ & \boxed{2} & \\ \boxed{3} & & \end{array}, \quad \text{then} \quad \widehat{S} = \begin{array}{ccc} \boxed{0} & \boxed{0} & \boxed{1} \\ \boxed{0} & \boxed{1} & \boxed{2} \\ \boxed{0} & \boxed{2} & \\ \boxed{3} & & \end{array} \quad \text{and} \quad \text{sign}(S) = (-1)^{18} = 1.$$

Remark 3.7.13. We consider the partition μ to be part of the data of S. For example, both $(1,1)/(1)$ and $(2)/(1)$ consist of a single box, but if we fill these two boxes with equal entries, the resulting skew tableaux have opposite signs.

Definition 3.7.14. Let μ, ν, λ be partitions. The *odd Littlewood-Richardson coefficient* $c_{\mu\nu}^{\lambda}$ is the coefficient of s_λ when $s_\mu s_\nu$ is expanded in the basis of odd Schur functions,

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^{\lambda} s_\lambda.$$

If $|\mu| + |\nu| \neq |\lambda|$, then $c_{\mu\nu}^{\lambda} = 0$.

Note that the odd Pieri rules compute certain odd Littlewood-Richardson coefficients. Using the involution $\psi_1\psi_2$ and the anti-involution ψ_3, we know the odd Littlewood-Richardson coefficient $c_{\mu\nu}^{\lambda}$ whenever μ or ν has either height or width 1.

If Y, Z are two nonzero monomials in Pol_n^{-1} such that $Y = \pm Z$, then let $\mathrm{sign}(Y, Z)$ denote the sign between them; for example $\mathrm{sign}(x_1x_2x_3, x_1x_3x_2) = -1$.

Lemma 3.7.15. The odd Littlewood-Richardson coefficient $c_{\mu\nu}^{\lambda}$ is

$$c_{\mu\nu}^{\lambda} = (-1)^{\binom{\mu}{2}+\binom{\nu}{2}+\binom{\lambda}{2}+N(\mu)+\sum_i (\lambda/\nu)_i \left|\frac{\nu}{i}\right|} \sum_{\substack{U \in SSYT(\mu) \\ UT_\nu = T_\lambda}} (-1)^{N^<(U)}. \tag{3.7.14}$$

In the summation condition, the product UT_ν is taken in the even plactic ring (so it is an equality up to sign in $\mathbb{Z}Pl_n$).

Proof. First note that

$$\begin{aligned} c_{\mu\nu}^{\lambda} &= (-1)^{\binom{\mu}{2}+\binom{\nu}{2}+\binom{\lambda}{2}+N(\mu)+N(\nu)+N(\lambda)} \sum_{\substack{U \in SSYT(\mu) \\ V \in SSYT(\nu) \\ UV = T_\lambda}} \mathrm{sign}(\widetilde{x}^{w_r(U)}\widetilde{x}^{w_r(V)}, \widetilde{x}^{w_r(T_\lambda)}) \\ &= (-1)^{\binom{\mu}{2}+\binom{\nu}{2}+\binom{\lambda}{2}+N(\mu)+N(\nu)+N(\lambda)} \sum_{\substack{U \in SSYT(\mu) \\ UT_\nu = T_\lambda}} \mathrm{sign}(\widetilde{x}^{w_r(U)}\widetilde{x}^{w_r(T_\nu)}, \widetilde{x}^{w_r(T_\lambda)}). \end{aligned} \tag{3.7.15}$$

The first equality is immediate from the definition of $c_{\mu\nu}^{\lambda}$ and equation (3.7.4). The second follows from the following fact: if U, V are semistandard tableaux of shapes μ, ν and $UV = T_\lambda$ in the even plactic ring, then $V = T_\nu$ [Ful97, Section 5.2]. To turn $\widetilde{x}^{w_r(U)}\widetilde{x}^{w_r(T_\nu)}$ into $\widetilde{x}^{w_r(T_\lambda)}$, we proceed in two steps:

1. Ascend-sort each of $\widetilde{x}^{w_r(U)}$, $\widetilde{x}^{w_r(T_\nu)}$, and $\widetilde{x}^{w_r(T_\lambda)}$ separately. This incurs the sign $(-1)^{N^<(U)+N(\nu)+N(\lambda)}$. The monomials are now $:\widetilde{x}^{w_r(T_{\lambda/\nu})}::\widetilde{x}^{w_r(T_\nu)}:$ and $:\widetilde{x}^{w_r(T_\lambda)}:$, where the colons denote ascend-sorting.

2. To sort these monomials together requires $\sum_i(\lambda/\nu)_i\left|\frac{\nu}{i}\right|$ transpositions.

The lemma follows. $\square$

Theorem 3.7.16 (Odd Littlewood-Richardson rule)**.** The odd Littlewood-Richardson coefficient $c_{\mu\nu}^\lambda$ is

$$c_{\mu\nu}^\lambda=(-1)^{N(\mu)+N(\lambda)}\sum_{\substack{S\in SSYT(\lambda/\mu,\nu)\\ w_r(S)\text{ is Yamanouchi}}}(-1)^{N^<(S)}. \tag{3.7.16}$$

Proof. The Littlewood-Richardson bijection of [Ful97, Section 5.2],

$$\{S\in SSYT(\lambda/\mu,\nu):w_r(S)\text{ is Yamanouchi}\}\to\{U\in SSYT(\mu,\lambda/\nu):UT_\nu=T_\lambda\}, \tag{3.7.17}$$

is described as follows. Let U_0 be a semistandard Young tableau of shape μ, all of whose entries are less than all of those of S. Then form $T_S\in SSYT(\lambda)$ by giving it the entries of U_0 on $\mu\subseteq\lambda$ and the entries of S on λ/μ. Under the RSK correspondence,

$$(T_\lambda,T_S)\leftrightarrow\binom{\underline{t}\,\underline{u}}{\underline{x}\,\underline{v}},$$

where $\underline{x},\underline{t}$ have length $|\mu|$ and $\underline{u},\underline{v}$ have length $|\nu|$. We can describe what these sub-words correspond to under the RSK correspondence as well:

$$(U,U_0)\leftrightarrow\binom{\underline{t}}{\underline{x}},\qquad(T_\nu,T_\nu)\leftrightarrow\binom{\underline{u}}{\underline{v}}$$

for some $U\in SSYT(\mu,\lambda/\nu)$. The Littlewood-Richardson bijection (3.7.17) assigns U to S.

In order to prove the theorem, we have to relate the quantities $N^<(U)$ and $N^<(S)$. The signed RSK correspondence (Theorem 2.2.16) allows us to do this; it implies

$$\begin{aligned}\operatorname{sign}(\underline{x}\,\underline{v})&=(-1)^{\binom{\lambda}{2}+N(\lambda)+N^<(T_S)}\\&=(-1)^{\binom{\lambda}{2}+N(\lambda)+N^<(U_0)+N^<(S)},\\\operatorname{sign}(\underline{x})&=(-1)^{\binom{\mu}{2}+N^<(U)+N^<(U_0)},\\\operatorname{sign}(\underline{v})&=(-1)^{\binom{\nu}{2}}.\end{aligned} \tag{3.7.18}$$

Since we know the contents of $\underline{x}$, $\underline{v}$, and $\underline{x}\,\underline{v}$, the sign of the latest can be expressed in terms of the former two,

$$\mathrm{sign}(\underline{x}\,\underline{v}) = \mathrm{sign}(\underline{x})\mathrm{sign}(\underline{v})(-1)^{\sum_i (\lambda/\nu)_i \left|\frac{\nu}{i}\right|}. \tag{3.7.19}$$

Comparing the signs in equations (3.7.18) and (3.7.19),

$$(-1)^{\binom{\mu}{2}+\binom{\nu}{2}+\binom{\lambda}{2}+N(\lambda)+N^{<}(U)+N^{<}(S)+\sum_i(\lambda/\nu)_i\left|\frac{\nu}{i}\right|} = 1.$$

Applying this to Lemma 3.7.15, the theorem follows. $\square$

Example 3.7.17. The first interesting cancellation is $c^{(3,2,1)}_{(2,1)(2,1)} = 0$ (in the even case, it equals 2). The Littlewood-Richardson skew tableaux of shape $(3,2,1)/(2,1)$ and content $(2,1)$ are

$$\begin{array}{ccc} & & \boxed{1} \\ & \boxed{1} & \\ \boxed{2} & & \end{array}, \qquad \begin{array}{ccc} & & \boxed{1} \\ & \boxed{2} & \\ \boxed{1} & & \end{array}.$$

They have $N^{<}(S)$ equal to 7 and 6, respectively.

Remark 3.7.18. It follows from Lemma 2.2.20 that

$$\begin{aligned} c^{\lambda}_{\mu\nu} &= (-1)^{\binom{\mu}{2}+\binom{\nu}{2}+\binom{\lambda}{2}+N(\mu)+N(\nu)+N(\lambda)} c^{\lambda}_{\nu\mu}, \\ c^{\lambda}_{\mu\nu} &= (-1)^{NE(\mu)+NE(\nu)+NE(\lambda)} c^{\lambda^T}_{\mu^T\nu^T}. \end{aligned} \tag{3.7.20}$$

These symmetries constrain some signs associated to Young diagrams. For instance, if $c^{\lambda}_{\mu\mu} \neq 0$, then $(-1)^{\binom{\lambda}{2}+N(\lambda)} = 1$.

3.7.4 Translation to Knutson-Tao hives

There are many combinatorial expressions for the even Littlewood-Richardson coefficients. Among them, the several which are expressible in terms of integer points of rational convex polytopes are especially interesting; one reason is that as result of Knutson and Tao's proof of the Saturation Conjecture [KT99], Klyachko's system of inequalities [Kly98] gives a necessary and sufficient criterion for a Littlewood-Richardson coefficient to be nonzero.

These expressions in terms of polytopes include Gelfand-Zeitlin (GZ) patterns [GZ86], Berenstein-Zelevinsky triangles [BZ92], the Littlewood-Richardson triangles of Pak and Vallejo [PV05], and the honeycombs and hives of Knutson and Tao [KT99]. As explained in the exposition of [Buc00], the hive model is particularly convenient and flexible. In this section, we will write down the bijection of [PV05] from Littlewood-Richardson skew tableaux to hives, by way of Littlewood-Richardson triangles. In both the triangle and hive settings, the sign associated to a diagram will come from a quadratic form on the ambient space of the polytope in question.

The following exposition follows [PV05] rather closely; our only contribution is to track the signs which arise in the odd case. For fixed $n \geq 1$, we will work in $V_{\mathbb{Z}} = \mathbb{Z}^{\binom{n+2}{2}}$ and $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$, with coordinates $\{a_{i,j} : 0 \leq i \leq j \leq n\}$. We call elements of V *triangles* and write them pictorially as (for $n = 3$)

$$\begin{array}{ccccccc} & & & a_{0,0} & & & \\ & & a_{0,1} & & a_{1,1} & & \\ & a_{0,2} & & a_{1,2} & & a_{2,2} & \\ a_{0,3} & & a_{1,3} & & a_{2,3} & & a_{3,3}. \end{array}$$

Definition 3.7.19. A *Littlewood-Richardson triangle* is an element $A = (a_{i,j}) \in V$ such that

1. $a_{i,j} \geq 0$ for $1 \leq i < j \leq n$,

2. $$\sum_{p=0}^{i-1} a_{p,j} \geq \sum_{p=0}^{i} a_{p,j+1} \qquad \text{for } 1 \leq i \leq j < n, \text{ and}$$

3. $$\sum_{q=i}^{j} a_{i,q} \geq \sum_{q=i+1}^{j+1} a_{i+1,q} \qquad \text{for } 1 \leq i < j < n.$$

Note that $a_{0,j} < 0$ and $a_{j,j} < 0$ are both possible.

Let $\triangle_{LR}$ denote the set of all Littlewood-Richardson triangles, a cone in V. Given a Littlewood-Richardson triangle $A = (a_{i,j}) \in \triangle_{LR}$, define partitions λ, μ, ν by

$$\lambda_j = \sum_{p=0}^{j} a_{p,j}, \qquad \mu_j = a_{0,j}, \qquad \nu_i = \sum_{q=i}^{k} a_{i,q}.$$

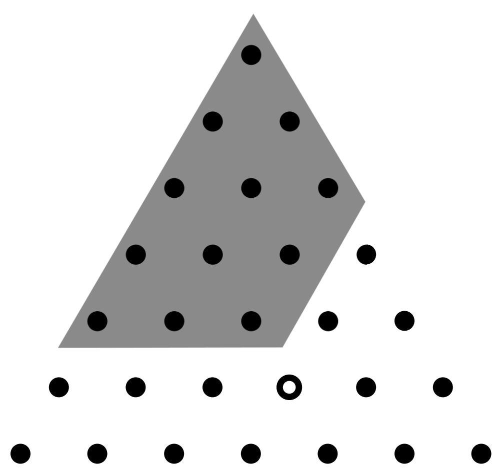
Let $\triangle_{LR}(\lambda,\mu,\nu)$ be the set of Littlewood-Richardson triangles with fixed λ,μ,ν. Each set $\triangle_{LR}(\lambda,\mu,\nu)$ is a convex polytope in V.

To a Littlewood-Richardson skew tableau $S \in SSYT(\lambda/\mu,\nu)$, associate an element A_S of $\triangle_{LR}(\lambda,\mu,\nu)$ by setting

$$\begin{aligned} &a_{0,0} = 0, \qquad a_{0,j} = \mu_j, \\ &a_{i,j} = \#\{\text{entries equal to } i \text{ in row } j \text{ of } S\} \quad (0 < i \le j). \end{aligned} \tag{3.7.21}$$

Lemma 3.7.20 ([PV05], Lemma 3.1)**.** Suppose $|\mu| + |\nu| = |\lambda|$. Then the assignment $S \mapsto A_S$ is a bijection between the set of Littlewood-Richardson tableaux in $SSYT(\lambda/\mu,\nu)$ and the set of integer points of $\triangle_{LR}(\lambda,\mu,\nu)$.

It is easy to read off $N^{<}(S)$ from the triangle $A_S = (a_{i,j})$. For each entry $a_{i,j}$, write $Y_{i,j}$ for the sum of all the $a_{p,q}$ in the shaded region below, where the (i,j) place is the dot drawn with a hollow center:



More formally,

$$Y_{i,j} = \sum_{p=0}^{i-1} \sum_{q=p}^{j-1} a_{p,q}. \tag{3.7.22}$$

Consider the quadratic form

$$Q_{\triangle}(A) = \sum_{i,j} a_{i,j} Y_{i,j} = \sum_{i=0}^{n} \sum_{j=i}^{n} \left(a_{i,j} \sum_{p=0}^{i-1} \sum_{q=p}^{j-1} a_{p,q} \right). \tag{3.7.23}$$

Then it is immediate from the description of the bijection above that

$$N^{<}(S) = Q_{\Delta}(A_S),$$

so

$$c_{\mu\nu}^{\lambda} = (-1)^{N(\mu)+N(\lambda)} \sum_{A \in \Delta_{LR}(\lambda,\mu,\nu)\cap V_{\mathbb{Z}}} (-1)^{Q_{\Delta}(A)}, \tag{3.7.24}$$

as long as λ, μ, ν all have at most n parts.

Example 3.7.21. If

$$S = \begin{array}{ccc} & & \boxed{1} \\ & \boxed{2} & \\ \boxed{1} & & \end{array},$$

then

$$A_S = \begin{array}{ccccccc} & & & 0 & & & \\ & & 2 & & 1 & & \\ & 1 & & 0 & & 1 & \\ 0 & & 1 & & 0 & & 0 \end{array}$$

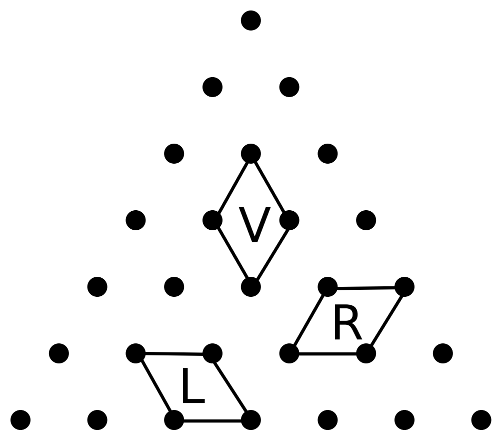
and $Q_{\Delta}(A_S) = 6$.

We now translate this result into the language of hives.

Definition 3.7.22. A *hive* is an element $H = (h_{i,j}) \in V$ with $h_{0,0} = 0$ which satisfies the inequalities

$$\begin{aligned} &(R) \qquad h_{i,j} - h_{i,j-1} \geq h_{i-1,j} - h_{i-1,j-1} \text{ for } 1 \leq i < j \leq n, \\ &(V) \qquad h_{i-1,j} - h_{i-1,j-1} \geq h_{i,j+1} - h_{i,j} \text{ for } 1 \leq i \leq j < n, \\ &(L) \qquad h_{i,j} - h_{i-1,j} \geq h_{i+1,j+1} - h_{i,j+1} \text{ for } 1 \leq i \leq j < n. \end{aligned} \tag{3.7.25}$$

Let $\mathfrak{H}$ be the set of all hives; this is a cone in V. The inequalities (3.7.25) have a geometric interpretation when we express H as a triangle. Inside a triangle, there are three types of rhombi which can be made out of two adjacent smallest-size triangles:



We call these right-slanted (R), vertical (V), and left-slanted (L) rhombi. The inequalities (3.7.25) say that the sum of the entries at the obtuse angles of any such rhombus is greater than or equal to the sum at the acute angles; the three inequalities are the right-slanted, vertical, and left-slanted cases, respectively.

As with Littlewood-Richardson triangles, we associate three partitions to a hive,

$$\lambda_j = h_{j,j} - h_{j-1,j-1}, \qquad \mu_j = h_{0,j} - h_{0,j-1}, \qquad \nu_i = h_{i,n} - h_{i-1,n}.$$

Let $\mathfrak{H}(\lambda,\mu,\nu)$ be the set of all hives with corresponding partitions λ,μ,ν. Each $\mathfrak{H}(\lambda,\mu,\nu)$ is a convex polytope in V.

Theorem 3.7.23 ([PV05], Theorem 4.1)**.** Let $\Phi : V \to V$ be the linear map which takes $A = (a_{i,j})$ to $H = (h_{i,j})$, where

$$h_{i,j} = \sum_{p=0}^{i}\sum_{q=p}^{j} a_{p,q}. \tag{3.7.26}$$

Then Φ is a volume-preserving isomorphism and induces bijections $\triangle_{LR}(\lambda,\mu,\nu) \to \mathfrak{H}(\lambda,\mu,\nu)$ for all λ,μ,ν.

As a matter of convention, let $h_{i,j} = 0$ if either $i > j$, $i < 0$, or $j > n$. It follows from equation (3.7.26) and an inclusion-exclusion argument that

$$\begin{aligned} &h_{i,j} = a_{i,j} + h_{i-1,j} + h_{i,j-1} - h_{i-1,j-1} \text{ if } 0 \le i < j \le n, \\ &h_{i,i} = a_{i,i} + h_{i-1,i} \text{ if } 0 \le i = j \le n. \end{aligned} \tag{3.7.27}$$

Let $Q_{\mathfrak{H}} = Q_{\Delta} \circ \Phi^{-1}$. Then for a hive $H = (h_{i,j})$, equations (3.7.27) imply

$$Q_{\mathfrak{H}}(H) = \sum_{i=1}^{n} \sum_{j=i}^{n} h_{i-1,j-1}\left(h_{i,j} - h_{i-1,j} - h_{i,j-1} + h_{i-1,j-1}\right) - \sum_{i=1}^{n-1} h_{i,i}^2. \tag{3.7.28}$$

Note that the parenthesized term is non-negative by the right-slanted rhombus inequality (R). It follows that if μ, ν, λ are partitions with at most n parts, then

$$N^{<}(S) = Q_{\Delta}(A_S) = Q_{\mathfrak{H}}(\Phi(A_S)), \tag{3.7.29}$$

so

$$c_{\mu\nu}^{\lambda} = (-1)^{N(\mu)+N(\lambda)} \sum_{H \in \mathfrak{H}(\lambda,\mu,\nu) \cap V_{\mathbb{Z}}} (-1)^{Q_{\mathfrak{H}}(H)}. \tag{3.7.30}$$

The quantity $N(\mu) + N(\lambda)$ can also be expressed as a quadratic form in either the variables $(a_{i,j})$ or the variables $(h_{i,j})$.

Example 3.7.24. With S and A_S as in Example 3.7.21,

$$\Phi(A_S) = \begin{array}{ccccccc} & & & 0 & & & \\ & & 2 & & 3 & & \\ & 3 & & 4 & & 5 & \\ 3 & & 5 & & 6 & & 6 \end{array} \tag{3.7.31}$$

and $Q_{\mathfrak{H}}(\Phi(A_S)) = 6$.

Chapter 4

Thick calculus and categorification

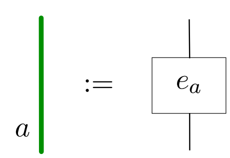
The results of this chapter, taken from [EKL12], are joint with Mikhail Khovanov and Aaron Lauda

4.1 Thick calculus

In this section we introduce an extension of the graphical calculus from the previous chapter.

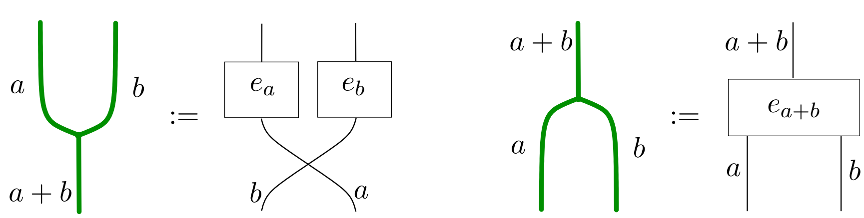
4.1.1 Splitters

Define



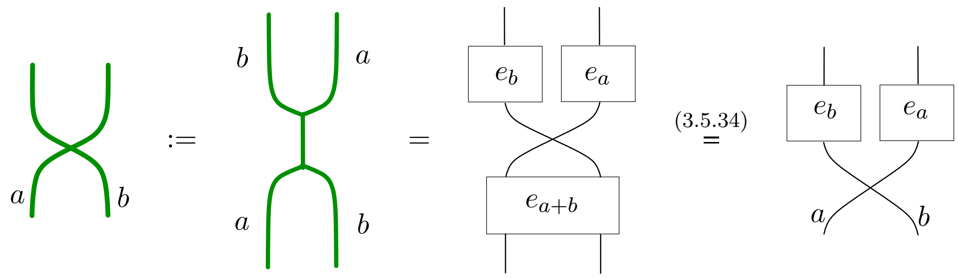
The notation is consistent, since e_a is an idempotent, so that cutting a thick line into two pieces and converting each of them into an e_a box results in the same element of the odd nilHecke ring.

The following diagrams will be referred to as *splitters* or splitter diagrams.



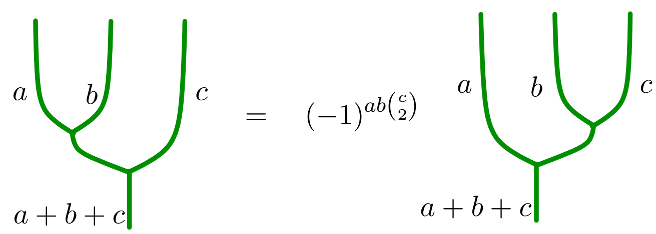
(4.1.1)

Define a thick crossing by

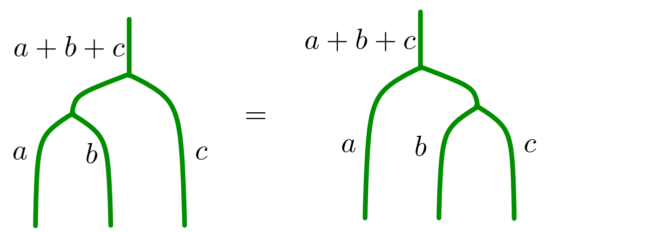


(4.1.2)

Proposition 4.1.1 (Associativity of splitters)**.**

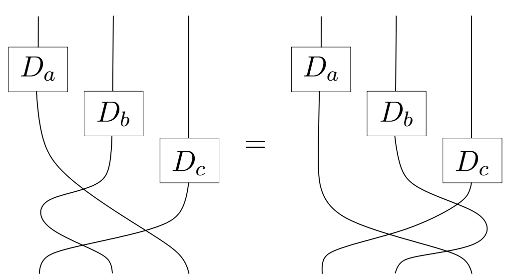


(4.1.3)



(4.1.4)

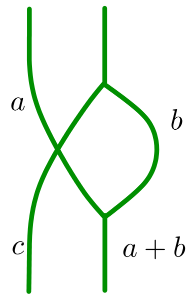
Proof. The proof of the first claim is a direct calculation making use of Proposition 3.5.5, equation 3.5.32, and the identity



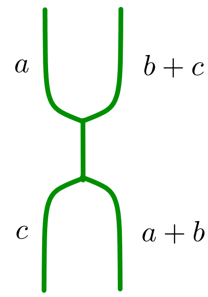
The second claim follows from equation (3.5.30). □

The apparent asymmetry between a, c in the splitter associativity relation is a consequence of the asymmetry in our choice of crossing between multiple strands, equation (3.5.31).

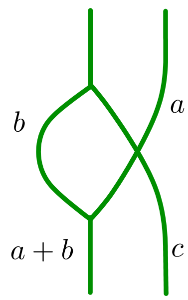
Proposition 4.1.2. For $a, b, c \geq 0$



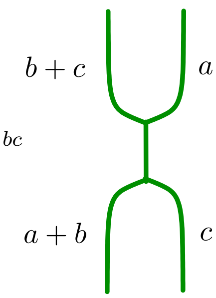
$=$



(4.1.5)



$= \quad (-1)^{\binom{a}{2}bc}$

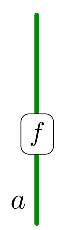


(4.1.6)

Proof. The proof uses equation 3.5.32 and equation (3.5.30). ☐

4.1.2 Dotted thick strands

Definition 4.1.3. For any $f \in O\Lambda_a$ write



$$:= \quad e_a f e_a. \tag{4.1.7}$$

For any polynomial $f \in \mathrm{Pol}_a^{-1}$ observe that $D_a f D_a = D_a(f) D_a$ in ONH_a. Now suppose f is odd symmetric. It follows that

$$e_a f e_a = \underline{x}^{\delta_a} D_a f \underline{x}^{\delta_a} D_a = \underline{x}^{\delta_a} D_a(f \underline{x}^{\delta_a}) D_a = (-1)^{\binom{a}{3}} \underline{x}^{\delta_a} f^{w_0} D_a. \tag{4.1.8}$$

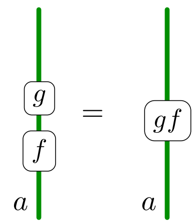
Then by the corollary to the OWL (3.2.29),

$$e_a f e_a = (-1)^{\binom{a}{3}} \underline{x}^{\delta_a} f^{w_0} D_a = e_a f. \tag{4.1.9}$$

It follows that

$$e_a g f e_a = (e_a g e_a)(e_a f e_a), \tag{4.1.10}$$

so we have the diagrammatic identity

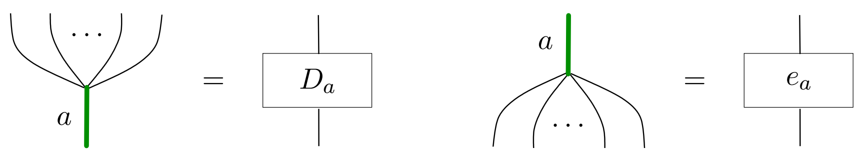


(4.1.11)

whenever $f, g \in O\Lambda_a$.

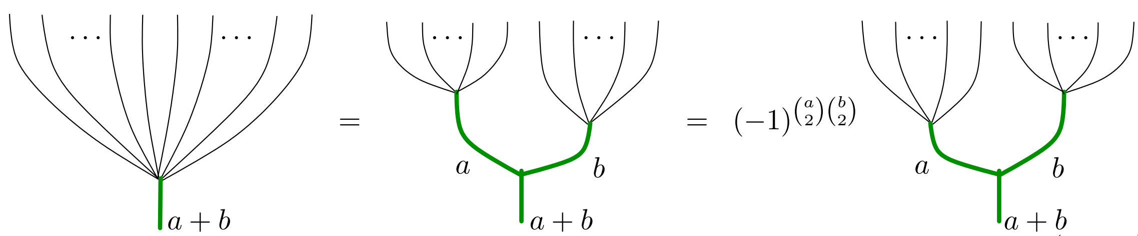
4.1.3 Explosions

Sometimes it is convenient to split thick edges into thin edges. Define *exploders*



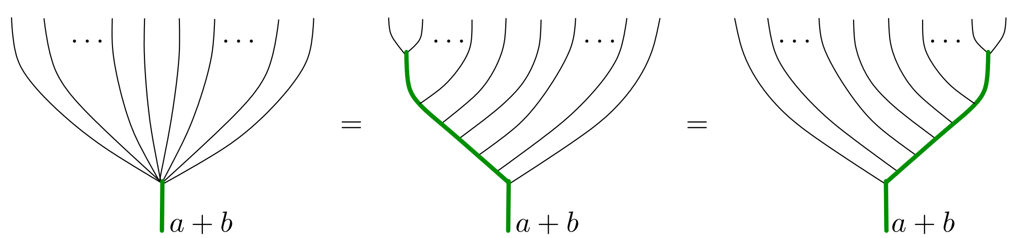
(4.1.12)

The associativity rules for exploded splitters follow immediately from the associativity rules for general splitters derived above. Equation (3.5.35) gives the following diagrammatic identities.



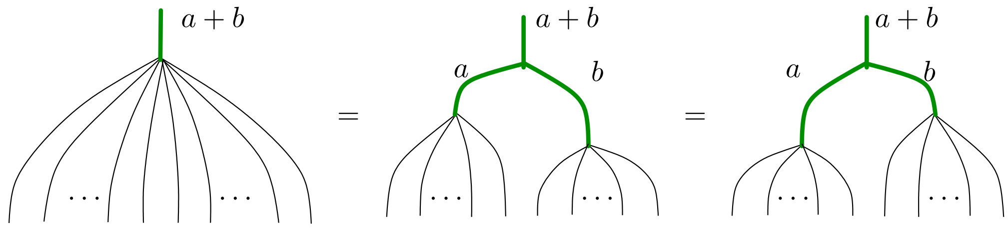
(4.1.13)

Using equation (3.5.34) the equation below follows.

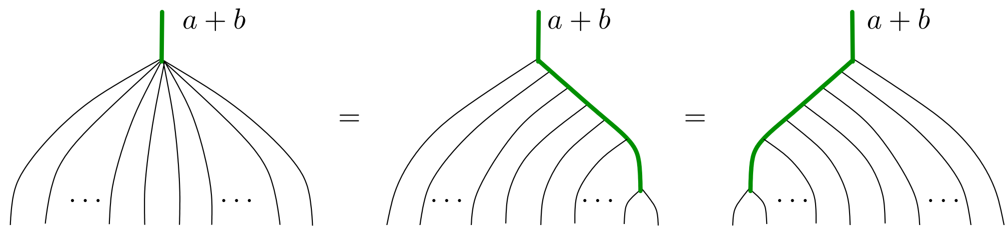


(4.1.14)

It is straightforward to show using the properties of the projectors e_a that the following relations hold,



(4.1.15)



(4.1.16)

4.1.4 Exploding Schur polynomials

Recall that given a partition $\alpha = (\alpha_1, \dots, \alpha_a) \in P(a)$ we have defined the odd Schur polynomial $s_\alpha(x_1, \dots, x_a)$ as

$$s_\alpha(x_1, \dots, x_a) := (-1)^{\binom{a}{3}} \left(D_a(\underline{x}^\alpha \underline{x}^{\delta_a}) \right)^{w_0}. \tag{4.1.17}$$

It will be convenient to normal order the variables using the notation

$$\underline{x}^{\delta_a + \alpha} = x_1^{a-1+\alpha_1} x_2^{a-2+\alpha_2} \dots x_a^{\alpha_a}. \tag{4.1.18}$$

One can show

$$s_\alpha(x_1, \dots, x_a) := (-1)^{\chi_\alpha^a} \left(D_a(\underline{x}^{\delta_a + \alpha}) \right)^{w_0}, \tag{4.1.19}$$

where for $\alpha \in P(a)$ we write

$$\chi_\alpha^a := \binom{a}{3} + |\alpha| \binom{a}{2} + \sum_{j=1}^{a} \alpha_j \binom{a-j+1}{2}. \tag{4.1.20}$$

That is, $(-1)^{\binom{a}{3}} \chi_\alpha^a$ is the sign needed to normal order the product $\underline{x}^\alpha \underline{x}^{\delta_a}$. For example, if $\alpha = (1^r)$ then $\alpha_j = 1$ for $1 \leq j \leq r$ and

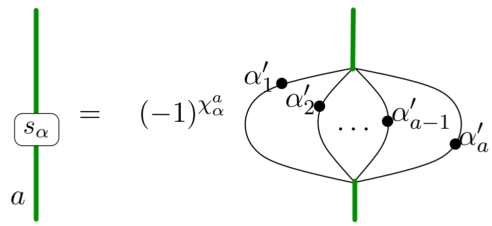
$$\chi_{(1^r)}^a = \binom{a}{3} + r\binom{a}{2} + \frac{r(3a^2 - 3ar + r^2 - 1)}{6}.$$

Observe that

$$e_a s_\alpha e_a = (-1)^{\chi_\alpha^a} e_a D_a(\underline{x}^{\alpha+\delta_a})^{w_0} e_a \overset{(4.1.9)}{=} (-1)^{\binom{a}{3} + \chi_\alpha^a} \underline{x}^{\delta_a} D_a(\underline{x}^{\alpha+\delta_a}) D_a \tag{4.1.21}$$

$$= (-1)^{\binom{a}{3} + \chi_\alpha^a} \underline{x}^{\delta_a} D_a \underline{x}^{\alpha+\delta_a} D_a \overset{(3.6.7)}{=} (-1)^{\chi_\alpha^a} e_a \underline{x}^{\alpha+\delta_a} D_a. \tag{4.1.22}$$

This implies the diagrammatic identity



(4.1.23)

where $\alpha'_j = \alpha_j + a - j$.

Notice that writing such an equation would not be possible if we did not include the action of the longest symmetric group element w_0 in the definition of s_α.

4.1.5 Orthogonal idempotents

4.1.5.1 Some helpful lemmas

Lemma 4.1.4 (Shuffle Lemma)**.**

$$\partial_i(x_i^m x_{i+1}^{m+k}) = \begin{cases} (-1)^m \partial_i(x_i^{m+k} x_{i+1}^m) & \text{if } k = 1, \\ -\partial_i(x_i^{m+k} x_{i+1}^m) & \text{if } k \text{ is even}, \\ (-1)^m \partial_i(x_i^{m+k} x_{i+1}^m) - \partial_i(x_i^{m+k-1} x_{i+1}^{m+1}) + (-1)^m \partial_i(x_i^{m+1} x_{i+1}^{m+k-1}) & \text{if } k \text{ is odd and } k \geq 3. \end{cases}$$

Proof. Using the Leibniz rule for odd divided difference operators we have

$$\partial_i(x_i^m x_{i+1}^{m+k}) = \partial_i(x_i^m x_{i+1}^m) x_{i+1}^k + (x_i^m x_{i+1}^m)^{s_i} \partial_i(x_{i+1}^k) = (x_{i+1}^m x_i^m) \partial_i(x_{i+1}^k). \qquad (4.1.24)$$

The sum $\widetilde{x}_i^k + \widetilde{x}_{i+1}^k$ is annihilated by the divided difference operator ∂_i when $k = 1$ and when k is even. When k is odd and $k \geq 3$, the sum

$$\widetilde{x}_i^k + \widetilde{x}_{i+1}^k + \widetilde{x}_i^{k-1} \widetilde{x}_{i+1} + \widetilde{x}_i \widetilde{x}_{i+1}^{k-1}, \qquad (4.1.25)$$

is annihilated by ∂_i. Using these facts the result follows. $\square$

Note that for k odd and $k \geq 3$ the relations above imply

$$\begin{aligned} &(-1)^m \partial_i(x_i^{m+k} x_{i+1}^m) - \partial_i(x_i^{m+k-1} x_{i+1}^{m+1}) + (-1)^m \partial_i(\widetilde{x}_i^{m+1} x_{i+1}^{m+k-1}) \\ &= (-1)^m \partial_i(x_i^{m+k} x_{i+1}^m) - 2 \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^{m(j+1)} \partial_i \left(x_i^{m+k-j} x_{i+1}^{m+j} \right) \end{aligned} \qquad (4.1.26)$$

so that

$$\partial_i(x_i^m x_{i+1}^{m+k}) = (-1)^m \partial_i(x_i^{m+k} x_{i+1}^m) - 2\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor}(-1)^{m(j+1)}\partial_i\left(x_i^{m+k-j}x_{i+1}^{m+j}\right). \qquad (4.1.27)$$

We call the relation above the *big odd shuffle*.

Proposition 4.1.5 (Vanishing of matched exponent in a partial staircase)**.**
Let $a \geq m \geq 2$ and $a-(m-1) \leq p \leq a-1$. Then

$$D_m(x_1^{a-1}x_2^{a-2}\dots x_{m-2}^{a-(m-2)}x_{m-1}^{a-(m-1)}x_m^p) = 0. \qquad (4.1.28)$$

Proof. We prove the proposition by induction on m. If $m=2$ we are done since then $p = a-1$ and $\partial_1(x_1^{a-1}x_2^{a-1}) = 0$. Assume by induction that

$$D_{m-1}(x_1^{a-1}x_2^{a-2}\dots x_{m-2}^{a-(m-2)}x_{m-1}^{p'}) = 0$$

for $a-(m-2) \leq p' \leq a-1$.

Suppose that $p = a-j$ for some $1 \leq j \leq m-1$ in the expression

$$D_m(x_1^{a-1}x_2^{a-2}\dots x_{m-1}^{a-(m-1)}x_m^p). \qquad (4.1.29)$$

If $p = a-(m-1)$ then we are done since $D_m = D_{m-1}(\partial_1 \dots \partial_{m-1})$ and $\partial_{m-1}(x_1^{a-1}\dots x_{m-1}^{a-(m-1)}x_m^{a-(m-1)}) = 0$. Assume then that $a-(m-2) \leq p \leq a-1$ so that

$$\begin{aligned} D_m(x_1^{a-1}\dots x_{m-1}^{a-(m-1)}x_m^p) &= D_{m-1}(\partial_1\dots\partial_{m-1})(x_1^{a-1}\dots x_{m-1}^{a-(m-1)}x_m^p) \\ &= D_{m-1}(\partial_1\dots\partial_{m-2})(x_1^{a-1}\dots x_{m-2}^{a-(m-2)}\partial_{m-1}(x_{m-1}^{a-(m-1)}x_m^p)). \end{aligned}$$

Let $q = p-(a-(m-1))$. Lemma 4.1.4 together with the induction hypothesis show that the expression vanishes whenever $q=1$ or q is even. When q is odd and $q \geq 3$ we write

$$\begin{aligned} &D_m(x_1^{a-1}\dots x_{m-1}^{a-(m-1)}x_m^p) = (-1)^{a-(m-1)}D_m(x_1^{a-1}\dots x_{m-2}^{a-(m-2)}x_{m-1}^p x_m^{a-(m-1)}) \\ &\quad -D_m(x_1^{a-1}\dots x_{m-2}^{a-(m-2)}x_{m-1}^{p-1}x_m^{a-(m-2)}) + (-1)^{a-(m-1)}D_m(x_1^{a-1}\dots x_{m-2}^{a-(m-2)}x_{m-1}^{a-(m-2)}x_m^{p-1}). \end{aligned}$$

The induction hypothesis shows that each of these three terms is zero, unless $p = a-(m-2)$ in which case the induction hypothesis does not apply to the second term. But in this case, the term still vanishes since the exponent of x_m is also $a-(m-2)$. $\square$

Proposition 4.1.6 (Adding a step to a full staircase)**.** The equation

$$D_a(x_1^{a-2}x_2^{a-3}\dots x_{a-1}^{0}x_a^{a-1}) = (-1)^{\binom{a-1}{3}} D_a(x_1^{a-1}x_2^{a-2}\dots x_{a-1}^{1}x_a^{0}) = (-1)^{\binom{a-1}{2}} \qquad (4.1.30)$$

holds in ONH_a.

Proof. The proof follows by shuffling the exponent of x_a left using Lemma 4.1.4 and the big odd shuffle. Additional terms that arise from the big odd shuffle vanish by Proposition 4.1.5. In this calculation, the sign is $(-1)^{\lfloor\frac{a-2}{2}\rfloor}$ when a is even, and $(a-1)$ when a is odd. ☐

Proposition 4.1.7 (Reordering a reverse staircase)**.**

$$D_a(x_1^0x_2^1\dots x_{a-1}^{a-2}x_a^{a-1}) = (-1)^{\binom{a}{4}} D_a(x_1^{a-1}x_2^{a-2}\dots x_{a-1}^{1}x_a^{0}). \qquad (4.1.31)$$

Proof. The proposition follows by induction from Proposition 4.1.6 together with some simplification of the exponent of (-1). ☐

Lemma 4.1.8. Let $\alpha \in P(a,b)$ and $\beta \in P(b,a)$ be two partitions. Then

$$D_{a+b}\left(x_1^{a-1+\alpha_1}x_2^{a-2+\alpha_2}\dots x_{a-1}^{1+\alpha_{a-1}}x_a^{\alpha_a}x_{a+1}^{\beta_b}x_{a+2}^{1+\beta_{b-1}}\dots x_{a+b-1}^{b-2+\beta_2}x_{a+b}^{b-1+\beta_1}\right) = 0 \qquad (4.1.32)$$

unless $|\alpha| + |\beta| = ab$.

Proof. If the total degree $\deg(\delta_a)+\deg(\delta_b)+|\alpha|+|\beta|$ of the dots is less than the degree $\binom{a+b}{2}$ of δ_{a+b}, then the total degree of the left-hand side of equation 4.1.33 is negative, and the expression must therefore be zero. If the degree is strictly greater than $\binom{a+b}{2}$, then using the Shuffle Lemma we can reorder the exponents so that they are decreasing, acquiring additional terms each time the big odd slide is performed. The requirements on α and β imply $b \geq \alpha_1 \geq \dots \geq \alpha_a$ and $a \geq \beta_1 \geq \dots \geq \beta_a \geq 1$. Since the total degree of the dots is greater than $\binom{a+b}{2}$ it follows that there must be a repeated exponent in each term that arises from the shuffling process. Since the action of D_{a+b} annihilates adjacent repeated exponents, all of these terms must vanish. ☐

Lemma 4.1.9. Let $\alpha \in P(a,b)$ and $\beta \in P(b,a)$. Then

$$D_{a+b}\left(x_1^{a-1+\alpha_1}x_2^{a-2+\alpha_2}\dots x_{a-1}^{1+\alpha_{a-1}}x_a^{\alpha_a}x_{a+1}^{\beta_b}x_{a+2}^{1+\beta_{b-1}}\dots x_{a+b-1}^{b-2+\beta_2}x_{a+b}^{b-1+\beta_1}\right) = \delta_{\alpha,\hat{\beta}}(-1)^{\Omega(\beta)+\binom{a+b}{3}}, \qquad (4.1.33)$$

where

$$\Omega(\beta) := \sum_{j=0}^{b-1} \binom{\beta_{b-j}+j}{3}. \qquad (4.1.34)$$

Proof. By the previous lemma it suffices to consider the case when $|\alpha| + |\beta| = ab$. The condition that $\alpha = \hat{\beta}$ implies that the only time (4.1.33) is nonzero is when each of the exponents is unique, so that all elements of the set $\{0, 1, \dots, a+b-1\}$ occur as exponents. Note that the partition requirements for α and β imply that there are chains of strict inequalities of exponents $a-1+\alpha_1 > a-2+\alpha_2 > \cdots > \alpha_a$ with the first a variables and $\beta_b < \beta_{b-1}+1 < \cdots < \beta_1 + b - 1$ with the last b variables.

We show that if an exponent is ever repeated in the left hand side of (4.1.33) then the expression vanishes. We will prove this in the case $\alpha_a < \beta_b$ (the case $\alpha_a > \beta_b$ is similar; and if $\alpha_a = \beta_b$, then the expression contains repeated adjacent exponents and is therefore zero). In this case, the exponents of $x_{a-\beta_b-1}$ through x_a must form a staircase

$$x_{a-\beta_b-1}^{\beta_b-1} x_{a-\beta_b+1}^{\beta_b-2} \dots x_{a-1}^{1} x_a^{0},$$

or else the expression vanishes. To see this suppose that x_{a-j} for $0 \leq j \leq \beta_b - 1$ is the first j where the exponent f of x_{a-j}^{f} is not part of a reverse staircase, that is, $f > j$.

Since $\beta_b > f$ all of the exponents of the last b strands must be larger than f, and all the variables before x_{a-j} must also be larger than f. Hence the exponent j does not occur in the expression

$$D_{a+b}\left(x_1^{a-1+\alpha_1} \dots x_{a-j-1}^{j+1+\alpha_{a-(j+1)}} x_{a-j}^{f} x_{a-(j-1)}^{(j-1)} \dots x_{a-1}^{1} x_a^{0} x_{a+1}^{\beta_b} \dots x_{a+b}^{b-1+\beta_1}\right).$$

Using the Shuffle Lemma and the big odd slide we reorder the exponents above so that they are decreasing. Each of the resulting terms will have one element of the set $\{0, 1, \dots, j-1\}$ missing since the shuffling procedure takes pairs of exponents (t_1, t_1+k) with $k > 0$ and either shuffles them (t_1+k, t_1) or else creates terms $(t_1+k-\ell, t_1+\ell)$ for $1 \leq \ell \leq \lfloor \frac{k}{2} \rfloor$. Observe that if a missing exponent appears from a big odd slide $(t_1+k-\ell, t_1+\ell)$, then the lower exponent t_1 has been removed. If t_1 then appears from a subsequent big odd slide, then again a lower exponent will have had to be removed. Hence, at least one element in the set $\{0, 1, \dots, j-1\}$ is missing in each term arising in the shuffling procedure. For degree reasons, if one exponent in the set $\{0, 1, \dots, a+b-1\}$ is missing, then at least one exponent must be repeated, hence the expression contains adjacent repeated exponents and must therefore vanish.

Thus, if our expression is nonvanishing, it must be of the form

$$D_{a+b}\left(x_1^{a-1+\alpha_1}\dots x_{a-\beta_b}^{\beta_b+\alpha_{(a-\beta_b)}} x_{a-(\beta_b-1)}^{\beta_b-1} x_{a-(\beta_b-2)}^{\beta_b-2}\dots x_{a-1}^{1} x_a^0 x_{a+1}^{\beta_b}\dots x_{a+b}^{b-1+\beta_1}\right).$$

Now use Proposition 4.1.6 to slide the exponent β_b of x_{a+1} left to add a step to the staircase giving the expression

$$(-1)^{\Omega(\beta_b)}D_{a+b}\left(x_1^{a-1+\alpha_1}\dots x_{a-\beta_b}^{\beta_b+\alpha_{(a-\beta_b)}} x_{a-(\beta_b-1)}^{\beta_b} x_{a-(\beta_b-2)}^{\beta_b-1}\dots x_{a-1}^{2} x_a^1 x_{a+1}^0 x_{a+2}^{1+\beta_{(b-1)}}\dots x_{a+b}^{b-1+\beta_1}\right).$$

If $\beta_{b-1}=\beta_b$, then slide the exponent $\beta_{b-1}+1=\beta_b+1$ of x_{a+2} left adding a step to the staircase with sign $(-1)^{\Omega(\beta_{b-1}+1)}$. If this exponent β_b+1 is repeated then it must be repeated as the exponent $\beta_b+\alpha_{(a-\beta_b)}$ of $x_{a-\beta_b}$, then the expression is zero. Otherwise we can assume that $\beta_{b-1}+1=\beta_b+1$ is not a repeated exponent and that $\beta_b+\alpha_{(a-\beta_b)}>\beta_{b-1}+1$.

If $\beta_{b-1}>\beta_b$ so that $\beta_{b-1}=\beta_b+g+1$ for some $g\geq 0$, then the staircase must continue

$$x_{a-(\beta_b+g)}^{\beta_b+g}\dots x_{a-(\beta_b+1)}^{\beta_b+2} x_{a-\beta_b}^{\beta_b+1} x_{a-(\beta_b-1)}^{\beta_b} x_{a-(\beta_b-2)}^{\beta_b-1}\dots x_a^1 x_{a+1}^0$$

or else the expression vanishes. If the staircase did not continue, then one of the exponents in the set $\{\beta_b+1,\beta_b+2\dots,\beta_b+g\}$ would not occur in the expression, and arguing as above one can show that all terms resulting from shuffling the expression to decreasing order must be missing at least one exponent in the set $\{0,1,\dots,\beta_b+g\}$ and must therefore vanish.

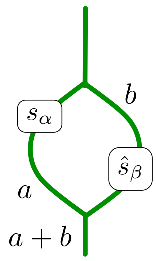
Continuing in this way, it follows that if any exponent in the expression is repeated then the expression is zero. Otherwise, all the exponents of the last b variables can be slid through staircases so that the resulting expression has strictly decreasing exponents. Since $D_{a+b}(\underline{x}^{a+b})=(-1)^{\binom{a+b}{3}}$ the result follows. □

Definition 4.1.10. Define *dual Schur functions* as

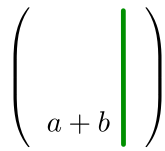
$$\hat{s}_\alpha(x_1,\dots,x_a):=(-1)^{\chi_\alpha^a}D_a(\sigma\psi(\underline{x}^{\delta_a+\alpha}))^{w_0}=(-1)^{\chi_\alpha^a}D_a(x_1^{\alpha_a}x_2^{1+\alpha_{a-1}}\dots x_a^{a-1+\alpha_1})^{w_0}.$$

It easy to see that $e_a\hat{s}_\alpha e_a=(-1)^{\chi_\alpha^a}e_a\sigma\psi(\underline{x}^{\delta_a+\alpha})D_a=(-1)^{\chi_\alpha^a}e_a x_1^{\alpha_a}x_2^{1+\alpha_{a-1}}\dots x_a^{a-1+\alpha_1}D_a$.

Proposition 4.1.11. Let $\alpha \in P(a,b)$ and $\beta \in P(b,a)$ be two partitions. Then

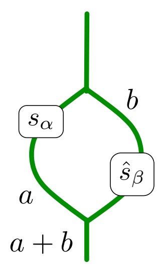


$$= \begin{cases} (-1)^{\chi_\alpha^a+\chi_{\hat\alpha}^b+\binom{a}{2}\left(|\hat\alpha|+\binom{b}{2}\right)+\Omega(\hat\alpha)+\binom{a+b}{3}} \left(\quad \right) & \text{if } \beta = \hat\alpha, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1.35}$$

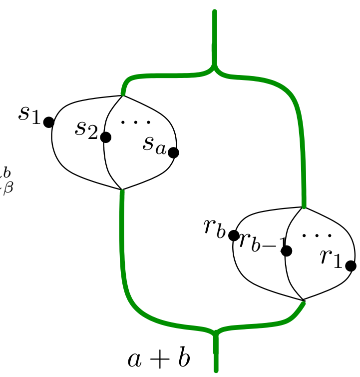


where $\Omega(\beta)$ is as in Lemma 4.1.9 and $\hat\alpha$ is as in Subsection 3.3.

Proof. From the definitions we have

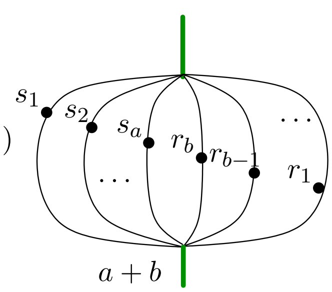


$$= (-1)^{\chi_\alpha^a+\chi_\beta^b} \tag{4.1.36}$$



where $s_j = a - j + \alpha_j$ and $r_j = b - j + \beta_j$. After sliding the splitters past dot terms and using associativity for exploded splitters in equations (4.1.13) and (4.1.15), the left-hand side of the above equation becomes

$$= (-1)^{\chi_\alpha^a+\chi_\beta^b}(-1)^{\binom{a}{2}\left(|\beta|+\binom{b}{2}\right)}$$



.

The result follows by Lemma 4.1.9. $\square$

Remark 4.1.12. The ordering of the x's in the above equation is critical for the above to work. Notice that this is a different ordering from what appears in [KLMS10]. To see why it is necessary consider the example with $a = 2$, $b = 2$, $\alpha = (2,0)$, and $\beta = (2,0)$. In this case, $D_4(x_1^3x_3^3) \neq 0$, while $D_4(x_1^3x_4^3) = 0$.

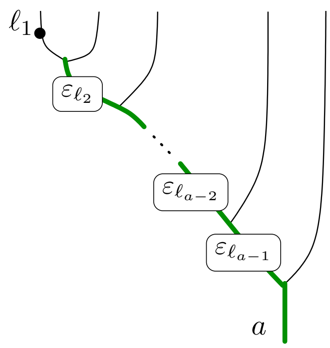
4.1.6 The odd nilHecke algebra as a matrix algebra

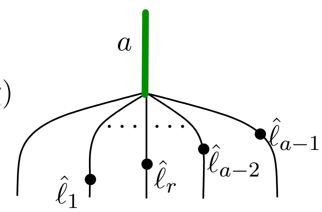
In this section we find it convenient to work with odd elementary symmetric functions ε_k rather than their Schur analogues $s_{(1^k)}$. Recall from (3.3.4) that the two are related by $s_{(1^k)} = (-1)^{\binom{k}{2}}\varepsilon_k$.

Let $\mathrm{Sq}(a)$ denote the set of sequences

$$\mathrm{Sq}(a) := \{\underline{\ell} = \ell_1 \dots \ell_{a-1} \mid 0 \leq \ell_\nu \leq \nu, \nu = 1, 2, \dots, a-1\}. \tag{4.1.37}$$

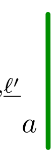
This set has size $|\mathrm{Sq}(a)| = a!$. For $\underline{\ell} \in \mathrm{Sq}(a)$ let

$\sigma_{\underline{\ell}} \quad =$ 

$\lambda_{\underline{\ell}} \quad = \quad (-1)^{\binom{a}{3}}$ 

(4.1.38)

Lemma 4.1.13.

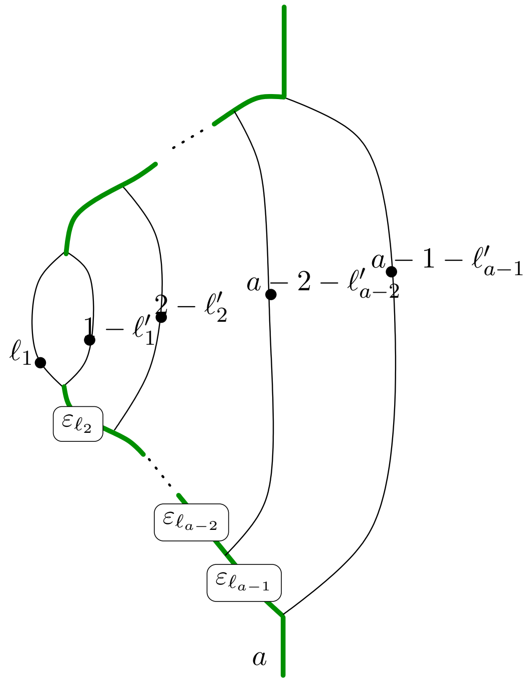
$\lambda_{\underline{\ell'}}\sigma_{\underline{\ell}} \quad = \quad \delta_{\underline{\ell},\underline{\ell'}}$ 

$= \quad \delta_{\underline{\ell},\underline{\ell'}} e_a$ (4.1.39)

for $\underline{\ell}$, $\underline{\ell'} \in \mathrm{Sq}(a)$.

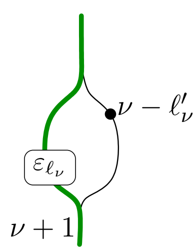
Proof. Consider the composite

$$\lambda_{\underline{\ell}'}\sigma_{\underline{\ell}} \quad = \quad (-1)^{\binom{a}{3}}$$

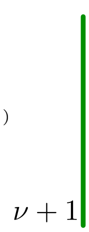


(4.1.40)

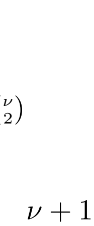
Repeatedly apply the equality



$$= \quad \delta_{\ell_\nu,\ell'_\nu}(-1)^{\binom{\ell_\nu}{2}+X^{\nu,1}_{(1^{\ell_\nu})}}$$



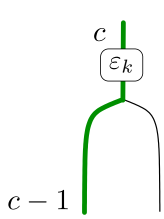
$$= \quad \delta_{\ell_\nu,\ell'_\nu}(-1)^{\binom{\nu}{2}}$$



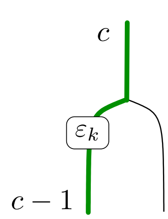
(4.1.41)

for $1 \leq \nu \leq a-1$, where the first equality follows from equation (3.3.4) and Proposition 4.1.11. The lemma follows since $\sum_{\nu=1}^{a-1}\binom{\nu}{2} = \binom{a}{3}$. □

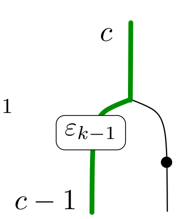
Using the definitions and equation (3.2.29) one can show that for $0 \leq k \leq a$ the diagrammatic identity



$$=$$



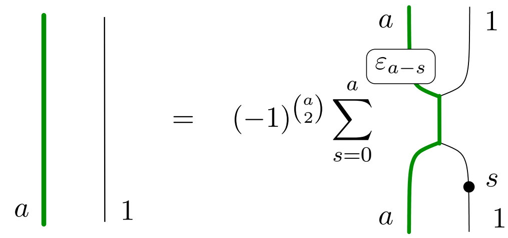
$$+ \quad (-1)^{c-1}$$



(4.1.42)

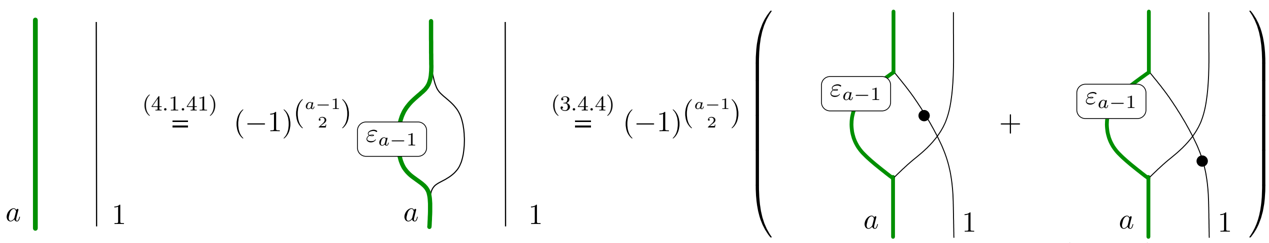
holds, where the first diagram on the right-hand side is zero by convention when $k = c$.

Lemma 4.1.14.



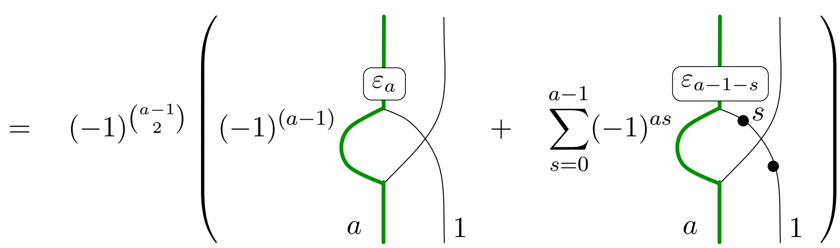
(4.1.43)

Proof. A direct computation gives



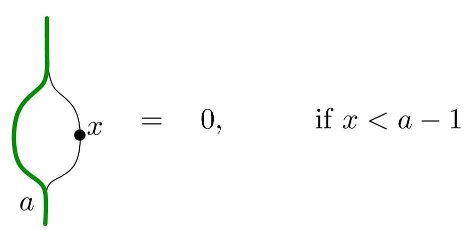
(4.1.44)

By (4.1.42) the odd elementary symmetric function can be slid to the top of the diagram.



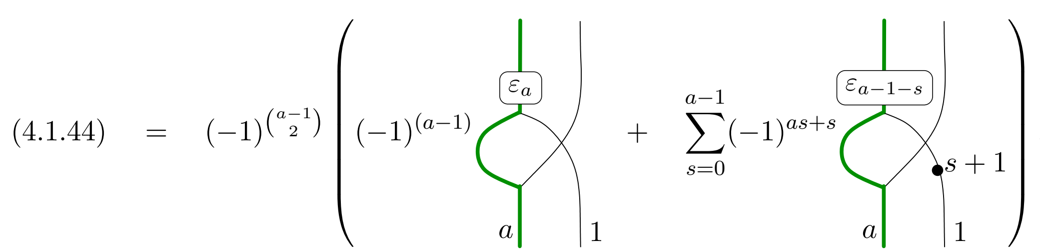
(4.1.45)

Sliding the s dots down on the second term using the odd nilHecke relation and the fact that



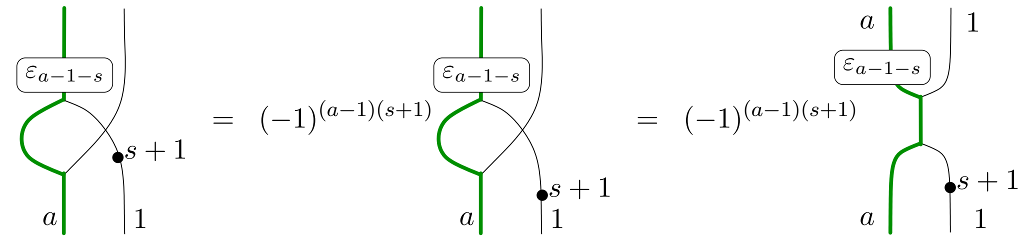
(4.1.46)

gives



(4.1.47)

In the last diagram on the right we can slide the $s+1$ dots down past the degree $a-1$ splitter giving

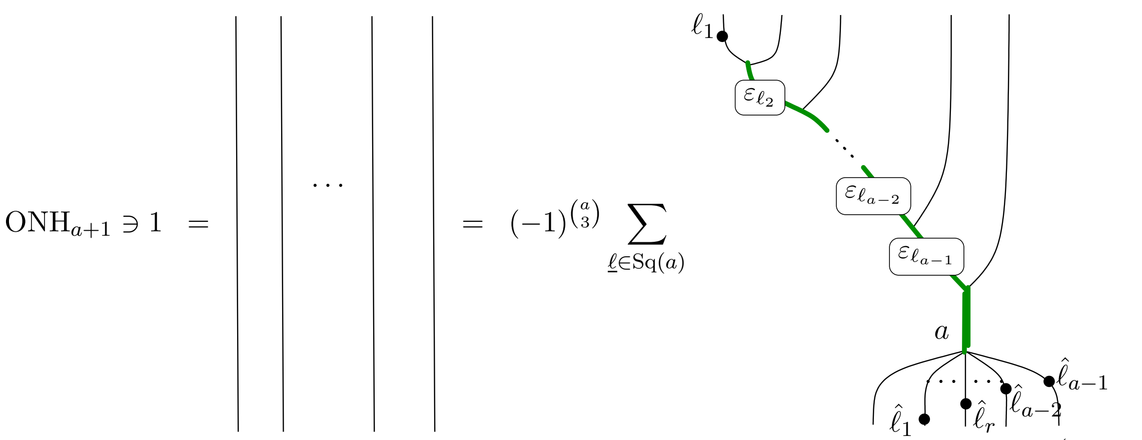


where we used Proposition 4.1.2 in the last equality. Shifting the summation shows that both terms in the parenthesis come with sign $(a-1)$. Since $(a-1)+\binom{a-1}{2}=\binom{a}{2}$ the result follows. $\square$

Theorem 4.1.15. Let $e_{\underline{\ell}} = \sigma_{\underline{\ell}}\lambda_{\underline{\ell}}$. The set $\{e_{\underline{\ell}}\}_{\underline{\ell}\in \mathrm{Sq}(a)}$ consists of mutually orthogonal idempotents that add up to $1 \in \mathrm{ONH}_a$:

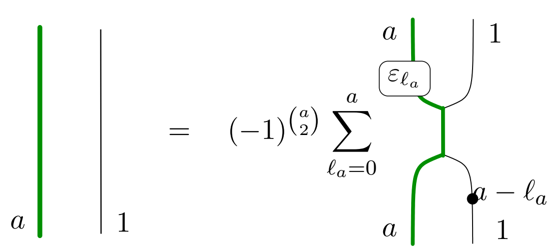
$$e_{\underline{\ell}}e_{\underline{\ell}'} = \delta_{\underline{\ell},\underline{\ell}'}e_{\underline{\ell}}, \qquad \sum_{\underline{\ell}\in \mathrm{Sq}(a)} e_{\underline{\ell}} = 1. \tag{4.1.48}$$

Proof. Lemma 4.1.13 shows that $e_{\underline{\ell}}$ are orthogonal idempotents. To see that they decompose the identity we proceed by induction, the base case being trivial.



(4.1.49)

then apply Lemma 4.1.14 in the form



(4.1.50)

and use that $\binom{a}{3} + \binom{a}{2} \equiv \binom{a+1}{3} \mod 2$, proving the inductive step. $\square$

Elements $\sigma_{\underline{\ell}}$, $\lambda_{\underline{\ell}}$ give an explicit realization of ONH_a as the algebra of $a! \times a!$ matrices over the ring of odd symmetric functions. Suppose that rows and columns of $a! \times a!$ matrices are enumerated by elements of $\mathrm{Sq}(a)$. The isomorphism takes the matrix with $x \in O\Lambda_a$ in the $(\underline{\ell}, \underline{\ell}')$ entry and zeros elsewhere to $\sigma_{\underline{\ell}} x \lambda_{\underline{\ell}'}$.

4.1.6.1 Decomposition of $\mathcal{E}^{(a)}\mathcal{E}^{(b)}$

Given $\alpha \in P(a,b)$ let

$$X_\alpha^{a,b} := |\alpha| \cdot |\hat{\alpha}| + \chi_\alpha^a + \chi_{\hat{\alpha}}^b + \binom{a}{2}\left(|\hat{\alpha}| + \binom{b}{2}\right) + \Omega(\hat{\alpha}) + \binom{a+b}{3}, \tag{4.1.51}$$

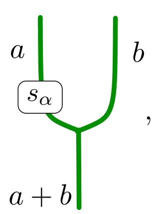
where Ω is defined by equation (4.1.34).

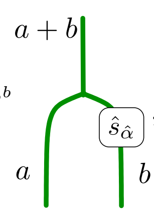
When $\alpha \in P(a,1)$, so that $\alpha = (1^{a-s})$ for some $0 \leq s \leq a$, then $|\alpha| = a - s$, $|\hat{\alpha}| = s$, and $X_\alpha^{a,1}$ from (4.1.51) simplifies to

$$X_{(1^r)}^{a,1} = a(a-r) + \binom{a-r+1}{2}. \tag{4.1.52}$$

In particular, $X_{(1^a)}^{a,1} = 1$.

For every partition $\alpha \in P(a,b)$ define

$\sigma_\alpha :=$ 

, $\lambda_\alpha := (-1)^{X_\alpha^{a,b}}$ 

, $e_\alpha = \sigma_\alpha \lambda_\alpha$. (4.1.53)

We view σ_α, λ_α, and e_α as elements of ONH_{a+b} with $\deg(\sigma_\alpha) = -\deg(\lambda_\alpha) = 2|\alpha| - 2ab$, and $\deg(e_\alpha) = 0$. Proposition 4.1.11 says that

$$\lambda_\beta \sigma_\alpha = \delta_{\alpha,\beta} e_{a+b}, \qquad \alpha, \beta \in P(a,b). \tag{4.1.54}$$

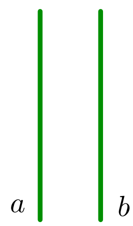
This implies

$$e_\beta e_\alpha = \delta_{\alpha,\beta} e_\alpha. \tag{4.1.55}$$

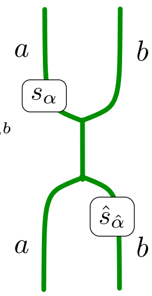
Theorem 4.1.16.

$$e_{a,b} = \sum_{\alpha \in P(a,b)} e_\alpha \tag{4.1.56}$$

or diagrammatically



$$= \sum_{\alpha \in P(a,b)} (-1)^{X_\alpha^{a,b}} \tag{4.1.57}$$



Proof. Proposition 4.1.11 implies that the e_α form a collection of $\binom{a+b}{a}$ mutually orthogonal idempotents in ONH_a. The result follows from the isomorphism $\mathrm{ONH}_a \cong \mathrm{Mat}_{(a)^!_{q^2}}(O\Lambda_a)$ in Corollary 3.2.5. □

4.2 Cyclotomic quotients

Let $a_1, a_2, \dots$ be a sequence of an algebra. We say this family satisfies the *odd symmetric defining relations* if

$$\begin{aligned} a_i a_j &= a_j a_j && \text{if } i+j \text{ is even,} \\ a_i a_j + (-1)^i a_j a_i &= (-1)^i a_{i+1} a_{j-1} + a_{j-1} a_{i+1} && \text{if } i+j \text{ is odd.} \end{aligned} \tag{4.2.1}$$

Usually, $a_i = 0$ for large enough i.

In terms of generators and relations, recall that the finite variable quotient of $O\Lambda$ is

$$O\Lambda_a \cong O\Lambda / \langle \varepsilon_m : m > a \rangle. \tag{4.2.2}$$

For each $N \geq 0$ we define the *odd Grassmannian ring* to be the quotient of $O\Lambda_a$ by the ideal generated by all h_m with $m > N - a$,

$$OH_{a,N} = O\Lambda_a / \langle h_m : m > N - a \rangle.$$

For $a = 0, N$ this ring is just $\mathbb{Z}$, and the it vanishes unless $0 \leq a \leq N$. The even analogue of $OH_{a,N}$ is the cohomology ring $H^*(\mathrm{Gr}(a, N))$ of the Grassmannian of complex a-planes in $\mathbb{C}^N$. The quotient $\mathrm{ONH}_a^N = \mathrm{ONH}_a / \langle x_1^N \rangle$ is called the N-th *cyclotomic quotient* of ONH_a.

Let $\mathcal{H}_a = \{\widetilde{x}^\alpha : \alpha \leq \delta\}$ be as in Lemma 3.2.3 and let $\varphi : \mathrm{ONH}_a \to \mathrm{End}_{O\Lambda_a}(\mathrm{Pol}_a^{-1})$ be the isomorphism of Corollary 3.2.5; under φ, a polynomial f acts as multiplication by f. A set of defining relations for the cyclotomic quotient ONH_a^N is given by the matrix entries of $\varphi(\widetilde{x}_1)^N$ with respect to the basis $\mathcal{H}_a$. We should therefore consider the operator $\varphi(\widetilde{x}_1)$ in some detail. Our analysis closely follows [Lau11, Section 5].

For each multi-index β obtained by replacing α_1 by zero in some α appearing in $\mathcal{H}_a$, consider the $O\Lambda_a$-submodule of Pol_a^{-1} with basis

$$B_\beta = \{\widetilde{x}_1^{a-1}\widetilde{x}^\beta, \widetilde{x}_1^{a-2}\widetilde{x}^\beta, \ldots, \widetilde{x}^\beta\}.$$

The operator $\varphi(\widetilde{x}_1)$ sends the span of each B_β to itself; let $\varphi(\widetilde{x}_1)_\beta$ be the resulting restricted map (or the corresponding matrix with respect to the basis B_β). So the defining relations given by the entries of $\varphi(\widetilde{x}_1)^N$ are all realized in the $a \times a$ matrices $\varphi(\widetilde{x}_1)_\beta^N$.

Lemma 4.2.1. The restriction of $\varphi(\widetilde{x}_1)$ to the span of B_β has the matrix expression

$$\varphi(\widetilde{x}_1)_\beta = \begin{pmatrix} \varepsilon_1 & 1 & 0 & 0 & \cdots & 0 \\ \varepsilon_2 & 0 & 1 & 0 & \cdots & 0 \\ -\varepsilon_3 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ (-1)^{\binom{a-2}{2}}\varepsilon_{a-1} & 0 & 0 & 0 & \cdots & 1 \\ (-1)^{\binom{a-1}{2}}\varepsilon_a & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{4.2.3}$$

with respect to the basis B_β.

Proof. Since $\widetilde{x}_1 \cdot \widetilde{x}_1^j\widetilde{x}^\beta = \widetilde{x}_1^{j+1}\widetilde{x}^\beta$, all entries other than the first column are clear. To get the first column, we need to express $\widetilde{x}_1^a\widetilde{x}^\beta$ in the basis B_β. That is, we want to find a linear relation

$$\widetilde{x}_1^a\widetilde{x}^\beta = f_1\widetilde{x}_1^{a-1}\widetilde{x}^\beta + \ldots + f_{a-1}\widetilde{x}_1\widetilde{x}^\beta,$$

with each $f_j \in O\Lambda_a$. Recall that $\varepsilon_0 = 1$. Now

$$\begin{aligned} \varepsilon_k(x_1, \ldots, x_a)\widetilde{x}_1^{a-k}\widetilde{x}^\beta &= \widetilde{x}_1\varepsilon_{k-1}(x_2, \ldots, x_a)\widetilde{x}_1^{a-k}\widetilde{x}^\beta + \varepsilon_k(x_2, \ldots, x_a)\widetilde{x}_1^{a-k}\widetilde{x}^\beta \\ &= \sum_{2\le i_1<\cdots<i_{k-1}\le a} \widetilde{x}_1\widetilde{x}_{i_1}\cdots\widetilde{x}_{i_{k-1}}\widetilde{x}_1^{a-k}\widetilde{x}^\beta + \sum_{2\le i_1<\cdots<i_k\le a} \widetilde{x}_{i_1}\cdots\widetilde{x}_{i_k}\widetilde{x}_1^{a-k}\widetilde{x}^\beta. \end{aligned}$$

Comparing this with the analogous expansion of $\varepsilon_{k+1}\widetilde{x}_1^{a-(k+1)}\widetilde{x}^\beta$, the first sum of the latter and the second sum of the former differ by the sign $(-1)^k$. So telescoping cancellations occur if each $f_j = \pm\varepsilon_j$ and $f_{j+1} = \pm(-1)^j\varepsilon_{j+1}$. The claim follows; that is, we have found

$$\widetilde{x}_1^a\widetilde{x}^\beta = \sum_{j=1}^{n}(-1)^{\binom{j-1}{2}}\varepsilon_j\widetilde{x}_1^{a-j}\widetilde{x}^\beta,$$

and that equation (4.2.3) is true. □

Proposition 4.2.2. The cyclotomic quotient ONH_a^N is isomorphic to a matrix algebra of size $q^{\binom{a}{2}}[a]!$ over the odd Grassmannian ring $OH_{a,N}$.

The proof will use an alternate presentation of $OH_{a,N}$. Let the algebra $O\Lambda_a[t]$ be obtained from $O\Lambda_a$ by adjoining an element t of super-degree 1 which is super-central; that is, $th_k = (-1)^k h_k t$ for all k (and likewise with h_k replaced by ε_k). The $\mathbb{Z}$-degree of t is immaterial, so set it to 2 for consistency with the x_i's. Then the relation

$$(1 + \varepsilon_1 t + \varepsilon_2 t^2 + \ldots)(1 + z_1 t + z_2 t^2 + \ldots) = 1 \tag{4.2.4}$$

holds if and only if $z_k = (-1)^{\binom{k+1}{2}}h_k$ for each k. So we can define $OH_{a,N}$ by taking the quotient of $O\Lambda_a$ by the ideal generated by the coefficients of powers t^k in

$$(1 + \varepsilon_1 t + \varepsilon_2 t^2 + \ldots + \varepsilon_a t^a)(1 + z_1 t + z_2 t^2 + \ldots + z_{N-a}t^{N-a}) = 1, \tag{4.2.5}$$

and furthermore we know that

$$z_k = (-1)^{\binom{k+1}{2}}h_k \tag{4.2.6}$$

for all k.

Proof. Let M be the matrix $\varphi(\widetilde{x}_1)_\beta$ of equation (4.2.3). By Lemma 4.2.1, the entries of M^N generate the ideal defining the cyclotomic quotient ONH_a^N. It is not hard to show that these relations are already generated by the entries in the first column of M^{N-a+1} (the proof is exactly as in the even case). These relations are homogeneous of degrees $2(N-a+1), 2(N-a+2), \ldots, 2N$. Let $v = (1, 0, \ldots, 0)^T$ be the column vector with first entry 1 and all other entries 0, so that we are seeking to compute the entries of $M^{N-a+1}v$. We proceed by induction on N. We may as well assume $N - a \geq a$; the case $N - a < a$ is similar. Let $f_{j,N-a}$ be the relation of degree $2(N-a+j)$ in equation (4.2.5), for each $j = 1, \ldots, a$. We claim that

$$(M^{N-a+1}v)_j = (-1)^{\binom{N-a+j+1}{2}}f_{j,N-a}.$$

For $N = a, a+1$, this is clear. Now replacing N by $N+1$, matrix multiplication shows

$$(M^{N-a+2}v)_j = \begin{cases} (-1)^{\binom{j-1}{2}+\binom{N-a}{2}}\varepsilon_j f_{1,N-a} + (-1)^{\binom{N-a+j+2}{2}} f_{j+1,N-a} & \text{for } 1 \leq j \leq a-1, \\ (-1)^{\binom{a-1}{2}+\binom{N-a+2}{2}}\varepsilon_a f_{1,N-a} & \text{for } j = a. \end{cases} \tag{4.2.7}$$

And by equation (4.2.5) with $N+1$ in place of N,

$$f_{j,N-a+1} = \begin{cases} f_{j+1,N-a} - (-1)^{j(N-a+1)}\varepsilon_j f_{1,N-a} & \text{for } 1 \leq j \leq a-1, \\ (-1)^{\binom{a-1}{2}+\binom{N-a+2}{2}}\varepsilon_a f_{1,N-a} & \text{for } j = a. \end{cases} \tag{4.2.8}$$

The expressions (4.2.7),(4.2.8) differ only by a sign, proving the proposition. □

In the even setting, the images of those Schur functions s_λ with $\lambda_1 \leq N-a$ and $\ell(\lambda) \leq a$ form an integral basis of $H^*(\mathrm{Gr}(a,N);\mathbb{Z})$; the class s_λ is Poincaré dual to the corresponding Schubert cycle.

Proposition 4.2.3. Over $\Bbbk = \mathbb{Z}$, the images of the Schur functions s_λ for λ having at most a rows and at most $N-a$ columns form a homogeneous basis for $OH_{a,N}$. All other Schur functions are 0 in $OH_{a,N}$.

Proof. It is shown in [EK12] that

$$s_\lambda^H = h_\lambda + \sum_{\mu > \lambda} a_\mu s_\mu^H,$$
$$s_\lambda^H = \varepsilon_{\overline{\lambda}} + \sum_{\mu > \overline{\lambda}} b_\mu s_\mu^H,$$

for certain integers a_μ, b_μ; the ordering on partitions here is the lexicographic one. As a result, if λ has more than a rows, s_λ^H is a linear combination of ε_μ's with $\mu_1 > a$ for all μ; hence $s_\lambda^H = 0$. Applying the involution $\psi_1\psi_2$, analogous reasoning with the odd complete polynomials implies that $s_\lambda^H = 0$ if λ has more than $N-a$ columns. So $OH_{a,N}$ is spanned over $\mathbb{Z}$ by the Schur functions s_λ^H for λ having at most a rows and at most $N-a$ columns. Reducing modulo 2, odd symmetric polynomials coincide with the usual (even) symmetric polynomials over $\mathbb{Z}/2$. In particular, they have the same graded rank over $\mathbb{Z}/2$. Since $OH_{a,N}$ is a free $\mathbb{Z}$-module, this implies that these Schur functions are in fact a basis over $\mathbb{Z}$. □

4.3 Categorification

Consider the category ONH_a-pmod of finitely generated graded left projective ONH_a-modules and its Grothendieck group $K_0(\mathrm{ONH}_a)$. We have a tower of superalgebras Grothendieck group

$$K_0(\mathrm{ONH}_\bullet) = \bigoplus_{a\geq 0} K_0(\mathrm{ONH}_a).$$

There is a natural inclusion of algebras $\mathrm{ONH}_a \otimes \mathrm{ONH}_b \subset \mathrm{ONH}_{a+b}$ given on diagrams by placing a diagram in ONH_a next to a diagram in ONH_b with the diagram from ONH_a appearing to the left of the one from ONH_b. These inclusions give rise to induction and restriction functors that equip $K_0(\mathrm{ONH}_\bullet)$ with the structure of a q-bialgebra.

Denote the regular representation of ONH_a by $\mathcal{E}^a$. By Corollary 3.2.5 this module decomposes into the direct sum of $a!$ copies of the unique indecomposable projective module of ONH_a. If we denote by $\mathcal{E}^{(a)}$ the projective module corresponding to the minimal idempotent e_a with the grading shifted down by $\binom{a}{2}$, then we get a direct sum decomposition of graded modules

$$\mathcal{E}^a = \mathrm{ONH}_a \cong \bigoplus_{[a]!} (\mathrm{ONH}_a e_a)\{-\binom{a}{2}\} = \bigoplus_{[a]!} \mathcal{E}^{(a)}.$$

Here $M^{\oplus f}$ or $\oplus_f M$, for a graded module M and a Laurent polynomial $f = \sum f_j q^j \in \mathbb{Z}[q, q^{-1}]$, denotes the direct sum over $j \in \mathbb{Z}$, of f_j copies of $M\{j\}$.

Diagrammatically, the projective module $\mathcal{E}^a$ corresponds to the idempotent $1 \in \mathrm{ONH}_a$ given by a vertical lines. The idempotent e_a corresponding to the indecomposable projective module $\mathcal{E}^{(a)}$ is represented in the graphical calculus by a thick edge of thickness a. Theorem 4.1.15 can be interpreted as giving an explicit isomorphism

$$\sum_{\underline{\ell}\in \mathrm{Sq}(a)} \lambda_{\underline{\ell}} : \mathcal{E}^a \longrightarrow \oplus_{[a]!}\mathcal{E}^{(a)} = \bigoplus_{\underline{\ell}\in \mathrm{Sq}(a)} \mathcal{E}^{(a)}\{a-1-2|\underline{\ell}|\}, \tag{4.3.1}$$

while Theorem 4.1.16 gives a canonical isomorphism

$$\sum_{\alpha\in P(a,b)} \lambda_\alpha \; : \; \mathcal{E}^{(a)}\mathcal{E}^{(b)}\mathbf{1}_n \longrightarrow \bigoplus_{\begin{bmatrix} a+b \\ a \end{bmatrix}} \mathcal{E}^{(a+b)} = \bigoplus_{\alpha\in P(a,b)} \mathcal{E}^{(a+b)}\mathbf{1}_n\{2|\alpha| - ab\} \tag{4.3.2}$$

In particular, we have an isomorphism

$$\begin{aligned} U_q^+(\mathfrak{sl}_2)_{\mathcal{A}} &\to K_0(\mathrm{ONH}_\bullet) \\ \theta^{(a)} &\mapsto [\mathcal{E}^{(a)}] \end{aligned} \tag{4.3.3}$$

where $U_q^+(\mathfrak{sl}_2)_{\mathcal{A}}$ is the integral version of the algebra $U_q^+(\mathfrak{sl}_2)$ (Definition 1.3.2). The same result was announced by Kang, Kashiwara, and Tsuchioka [KKT11].

Denote by $\mathrm{ONH}^N := \bigoplus_{a=0}^N \mathrm{ONH}_a^N$ the direct sum of cyclotomic quotients of ONH_a. The odd Grassmannian ring $OH_{a,N}$ is graded local, so Proposition 4.2.2 implies that $K_0(\mathrm{ONH}_\bullet^N)$ has the same size as the integral form of the irreducible representation of $U_q(\mathfrak{sl}_2)$ of highest weight N (the cyclotomic quotient ONH_a^N is zero unless $0 \leq a \leq N$). The tensor products by bimodules which are odd analogues of the cohomology of two-step flag varieties descend to the action of E and F on $K_0(\mathrm{ONH}_\bullet^N)$, as in the even case.

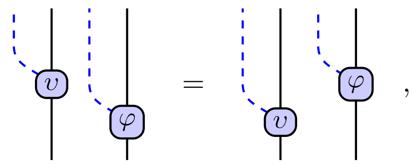
4.4 More categorification

4.4.1 Further results on the odd 2-representation theory of $U_q(\mathfrak{sl}_2)$

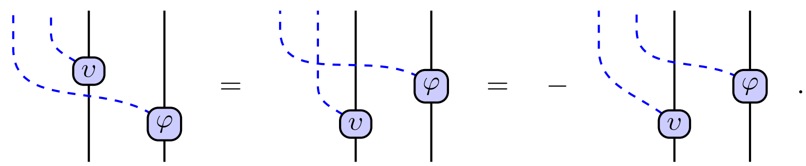
Since the results of this thesis originally appeared, there has been progress on odd categorification. Here is a summary of recent progress:

Decategorified	Categorified	Reference
$U_q^+(\mathfrak{sl}_2)$	odd nilHecke algebras	[EKL12]
$U_q^+(\mathfrak{g})$	quiver Hecke superalgebras	[HW12; KKT11]
integrable highest weight simples	cyclotomic quotients	[KKO12; KKO13]
tensor products of simples	odd Webster algebras	*in preparation*
$U_q(\mathfrak{sl}_2)$ and its canonical basis	"odd $\dot{\mathcal{U}}$" and its indecomposable projectives	*in preparation*
$U_{\sqrt{-1}}(\mathfrak{sl}_2)$, simples, and tensor products	a dg-structure on the above	*in preparation*

In order to realize an odd categorification of the full quantum group $U_q(\mathfrak{sl}_2)$, an appropriate super-2-categorical setting is required. It is an interesting side effect of this that the skew commutativity of the odd nilHecke algebra is replaced by a genuinely isotopy-invariant diagrammatics. The following picture may prove more illuminating than a list of axioms:



(4.4.1)



(4.4.2)

In the above, υ and φ are odd-parity 2-morphisms (think of dots in the odd nilHecke algebra), and a dashed blue line represents a parity shift (generator of the $\mathbb{Z}/2$-action). A diagram containing a dashed-dashed crossing equals -1 times the same diagram with the crossing resolved.

As described in the Introduction, one goal of the above is to give a 2-representation theoretic construction of odd Khovanov homology [ORS07].

4.4.2 The covering setting

If an additive category $\mathcal{C}$ has a G-grading, then its Grothendieck group $K_0(\mathcal{C})$ is naturally a $\mathbb{Z}[G]$-module. For this case $G = \mathbb{Z}$, this is precisely the source of the $\mathbb{Z}[q, q^{-1}]$ structure in the Khovanov-Lauda categorification of $U_q^+(\mathfrak{g})$ for generic q. In the odd setting, the super-structure leads to an additional $\mathbb{Z}/2$-grading. In our categorification of $U_q^+(\mathfrak{sl}_2)$ above, this resulting $\mathbb{Z}[\mathbb{Z}/2]$-action is trivial.

In [HW12], David Hill and Weiqiang Wang noticed that having $\mathbb{Z}[\mathbb{Z}/2]$ act by the sign representation gives an isomorphism between Grothendieck group $\bigoplus_{n\geq 0} K_0(\mathrm{ONH}_n)$ and $U_q^+(\mathfrak{osp}_{1|2})_{\mathcal{A}}$ rather than $U_q^+(\mathfrak{sl}_2)_{\mathcal{A}}$. The Lie superalgebra $\mathfrak{osp}_{1|2}$ is obtained from $\mathfrak{sl}_2$ by "adjoining square roots of E and F"; see [HW12] for details and the quantum deformation. They also generalize this construction to certain other quantum Kac-Moody algebras having no odd isotropic roots, using the quiver Hecke superalgebras of [KKT11]. In fact, they introduce a quantum Kac-Moody algebra defined over $\mathbb{Z}[\pi]/(\pi^2 - 1)$ called the *covering quantum Kac-Moody algebra* for $\mathfrak{sl}_2$. This algebra specializes to $U_q(\mathfrak{sl}_2)$ when $\pi = 1$ and to $U_q(\mathfrak{osp}_{1|2})$ when $q = -1$. The odd nilHecke algebra categorifies this covering algebra.

The result, a categorification of both $U_q^+(\mathfrak{sl}_2)$ and $U_q^+(\mathfrak{osp}_{1|2})$ by the same algebras, is a genuinely odd phenomenon; the analogous even case construction does not work because $\mathrm{NH}_a \otimes \mathrm{NH}_b \hookrightarrow \mathrm{NH}_{a+b}$ is a homomorphism of algebras but not as superalgebras. There is a lot of interesting work to be done in exploring the ramifications of this "covering setting." This is a higher manifestation of the idea that remembering even and odd quantities simultaneously—Kostka numbers, Littlewood-Richardson coefficients, bilinear form coefficients, and so forth—gives greater rigidity than can illuminate the way to higher structure.

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Appendix A

Data

As we are introducing several new combinatorial objects, it seems appropriate to provide some data for others to work with. Python code for working with q-polynomials, odd symmetric functions, and even and odd nilHecke algebras is freely and publicly available from the author's Github account at `https://github.com/apellis`. The relevant repositories are `symmetric`, `nc_polynomial`, `odd_code`, and `manin_schechtman`. Python 2.x and the `numpy` package are required for everything; the package `networkx` is used by `odd_code` and `manin_schechtman` and the package `matplotlib` is used in visualizations.

A.1 The bilinear form

All matrices in this subsection are in the complete basis, ordered lexicographically among compositions (for unspecified q) or partitions (for $q = -1$). As these matrices are symmetric, we use $*$'s to abbreviate redundant entries.

A.1.1 Unspecified q

deg. 1	h_1
h_1	1

deg. 2	h_{11}	h_2
h_{11}	$[2]$	1
h_2	$*$	1

deg. 3	h_{111}	h_{12}	h_{21}	h_3
h_{111}	$[3]!$	$[3]$	$[3]$	1
h_{12}	$*$	$[2]$	$1+q^2$	1
h_{21}	$*$	$*$	$[2]$	1
h_3	$*$	$*$	$*$	1

deg. 4	h_{1111}	h_{112}	h_{121}	h_{211}	h_{22}	h_{13}	h_{31}	h_4
h_{1111}	$[4]!$	$[4][3]$	$[4][3]$	$[4][3]$	$[5]+q^2$	$[4]$	$[4]$	1
h_{112}	$*$	$[5]+q[2]$	$[5]+q^2[2]$	$[6]+q^2$	$[3]+q^4$	$[3]$	$[1]+q^2[2]$	1
h_{121}	$*$	$*$	$[4]+q+q^3+q^5$	$[5]+q^2[2]$	$1+2q^2+q^3$	$[2]+q^3$	$[2]+q^3$	1
h_{211}	$*$	$*$	$*$	$[5]+q[2]$	$[3]+q^4$	$1+q^2[2]$	$[3]$	1
h_{22}	$*$	$*$	$*$	$*$	$[2]+q^4$	$1+q^2$	$1+q^2$	1
h_{13}	$*$	$*$	$*$	$*$	$*$	$[2]$	$1+q^3$	1
h_{31}	$*$	$*$	$*$	$*$	$*$	$*$	$[2]$	1
h_4	$*$	$*$	$*$	$*$	$*$	$*$	$*$	1

A few remarks on the determinant of the bilinear form are in order. It is immediate from the definition of the bilinear form that this determinant is monic in q. It is not hard to see that its degree is given by

$$\begin{aligned} D &= \sum_{\alpha}\left(\frac{1}{2}n(n-1) - \sum_{i=1}^{\ell(\alpha)}\frac{1}{2}\alpha_i(\alpha_i-1)\right) \\ &= 2^{n-2}n(n-1) - \frac{1}{2}\sum_{\alpha}\sum_{i=1}^{\ell(\alpha)}\alpha_i(\alpha_i-1). \end{aligned} \tag{A.1.1}$$

The outer summation is over all decompositions α of n, and $\ell(\alpha)$ is the length of the decomposition α. Let

$$A_n = \sum_{\alpha}\sum_{i=1}^{\ell(\alpha)}\alpha_i(\alpha_i-1)$$

be the double summation of equation (A.1.1). Re-indexing so as to first sum over the first entry of each decomposition, we see

$$A_n = n(n-1) + \sum_{k=1}^{n-1}\left[2^{n-k-1}k(k-1) + A_{n-k}\right].$$

This recursion can be solved as

$$A_n = 2 + 2^n(n-2),$$

from which we conclude that the degree of the determinant of the bilinear form, as a polynomial in

q, equals

$$D = 2^{n-2}\left(n^2 - 3n + 4\right) - 1. \tag{A.1.2}$$

A.1.2 $q = -1$

deg. 1	h_1
h_1	1

deg. 2	h_{11}	h_2
h_{11}	0	1
h_2	1	1

deg. 3	h_{111}	h_{21}	h_3
h_{111}	0	1	1
h_{21}	1	0	1
h_3	1	1	1

deg. 4	h_{1111}	h_{211}	h_{22}	h_{31}	h_4
h_{1111}	0	0	2	0	1
h_{211}	0	1	2	1	1
h_{22}	2	2	1	2	1
h_{31}	0	1	2	0	1
h_4	1	1	1	1	1

deg. 5	h_{11111}	h_{2111}	h_{221}	h_{311}	h_{32}	h_{41}	h_5
h_{11111}	0	0	2	0	2	1	1
h_{2111}	0	1	0	1	3	0	1
h_{221}	2	0	-3	2	3	-1	1
h_{311}	0	1	2	1	2	1	1
h_{32}	2	3	3	2	1	2	1
h_{41}	1	0	-1	1	2	0	1
h_5	1	1	1	1	1	1	1

deg. 6	h_{111111}	h_{21111}	h_{2211}	h_{222}	h_{3111}	h_{321}	h_{33}	h_{411}	h_{42}	h_{51}	h_6
h_{111111}	0	0	0	6	0	0	0	0	3	0	1
h_{21111}	0	0	2	6	0	2	2	1	3	1	1
h_{2211}	0	2	4	3	2	4	4	2	2	2	1
h_{222}	6	6	3	-3	6	5	5	3	0	3	1
h_{3111}	0	0	2	6	0	-1	0	1	3	0	1
h_{321}	0	2	4	5	-1	-4	-2	2	3	-1	1
h_{33}	0	2	4	5	0	-2	0	2	3	0	1
h_{411}	0	1	2	3	1	2	2	1	2	1	1
h_{42}	3	3	2	0	3	3	3	2	1	2	1
h_{51}	0	1	2	3	0	-1	0	1	2	0	1
h_6	1	1	1	1	1	1	1	1	1	1	1

A.2 Bases of $O\Lambda$

A.2.1 Elementary functions

We list the first few odd elementary functions. As described in Corollary 2.2.4, they form a multiplicative basis.

$$
\begin{aligned}
e_1 &= h_1, \\
e_2 &= h_2 - h_1^2, \\
e_3 &= h_3 - h_1^3, \\
e_4 &= -h_4 + h_2^2 - h_2 h_1^2 + h_1^4, \\
e_5 &= h_5 - 2h_4 h_1 - h_3 h_1^2 + h_2^2 h_1 + h_1^5.
\end{aligned}
$$

A.2.2 Power sum functions

Though they do not generate a multiplicatie basis, we list the first few odd power sum functions.

$$p_1 = h_1,$$
$$p_2 = h_{11},$$
$$p_3 = h_{111} + h_{21} - h_3,$$
$$p_4 = -h_{1111} - 2h_{22} + 4h_4,$$
$$p_5 = h_{11111} + h_{2111} + 3h_{221} - h_{311} - 3h_{32} - 9h_{41} + 9h_5,$$
$$p_6 = h_{111111} + 3h_{2211} - 3h_{33} - 6h_{411} + 6h_{51}.$$

A.2.3 Schur functions

$$s_1 = h_1$$
$$s_{11} = h_{11} - h_2$$
$$s_2 = h_2$$
$$s_{111} = h_{111} - h_3$$
$$s_{21} = h_{21} - h_3$$
$$s_3 = h_3$$
$$s_{1111} = h_{1111} - h_{211} + h_{22} - h_4$$
$$s_{211} = h_{211} - h_{22} - h_{31} + h_4$$
$$s_{22} = h_{22} + h_{31} - 2h_4$$
$$s_{31} = h_{31} - h_4$$
$$s_4 = h_4$$
$$s_{11111} = h_{11111} + h_{221} - h_{311} - 2h_{41} + h_5$$
$$s_{2111} = h_{2111} - h_{311} - h_{41} + h_5$$
$$s_{221} = h_{221} + h_{311} - h_{32} - 3h_{41} + 2h_5$$
$$s_{311} = h_{311} - h_{32} - h_{41} + h_5$$
$$s_{32} = h_{32} + h_{41} - 2h_5$$
$$s_{41} = h_{41} - h_5$$
$$s_5 = h_5$$